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Exact Renormalization Group and Loop Variables: A Background Independent Approach to String Theory.

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Abstract

This paper is a self contained review of the Loop Variable approach to string theory. The Exact Renormalization Group is applied to a world sheet theory describing string propagation in a general background involving both massless and massive modes. This gives interacting equations of motion for the modes of the string. Loop variable techniques are used to obtain gauge invariant equations. Since this method is not tied to flat space time or any particular background metric, it is manifestly background independent. The technique can be applied to both open and closed strings. Thus gauge invariant and generally covariant interacting equations of motion can be written for massive higher spin fields in arbitrary backgrounds. Some explicit examples are given.

Contents

1 Introduction

This is a review article on the Loop Variable (LV) approach to obtaining equations of motion (EOM) for the various modes of the string. The LV method is based on the old idea [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] that requiring conformal invariance of the Polyakov action describing strings propagating in a background gives EOM for the background fields. From the point of view of the Polyakov action, these background fields are generalized coupling constants and their EOM are just generalized β functions. This technique was first applied to massless fields in the closed string where Einstein's equations were obtained. Similarly for open strings Maxwell's equations are obtained to lowest order in α'. At higher orders one obtains Dirac-Born-Infeld type generalization of Maxwell's action[12]. The vertex operators describing massless gauge fields are marginal operators near zero momentum. So the beta functions do indeed give low energy equations. For a massive field such as the tachyon, vertex operators become marginal for non zero momentum (near the mass shell). The beta function calculation is a little more complicated [6] for massive modes. A related proper time formalism was developed in [13]. It can be shown [14, 13, 16, 17] that to all orders the equation of motion so obtained is equivalent to the S-matrix for on shell tachyons.[1] When we get to the higher spin massive modes the naive method breaks down. This is because the equations obtained by this method are not gauge invariant.

The problem of obtaining gauge invariant EOM is addressed by the LV formalism. A key idea in this formalism is borrowed from BRST string field theory where it was shown that including the extra fields needed for a gauge invariant description of string theory requires an extra world sheet field - the bosonized ghost coordinate. In the LV formalism it is found that the theory looks massless in one extra dimension and mass is obtained as the extra momentum component during dimensional reduction. The other key ingredient is the introduction of an infinite number of parameters - a generalization of the proper time - to parametrize the space of gauge transformations. These extra proper time parameters are reminiscent of the time parameters in the KP hierarchy. There too the group $Diff(S^1)$, parametrized by an infinite number of variables plays an important role. A third ingredient is that all the vertex operators are collected together in one non local object - the loop variable. Thus unlike in string field theory where one expands in oscillators, here the expansion is in terms of vertex operators. All these ingredients put together give us a way of writing a very general two dimensional field theory. Instead of calculating the beta function for marginal operators as one does in a continuum field theory, here we apply the Exact Renormalization Group (ERG) transformation as first defined by Wilson [19, 20, 21, 22], on this cutoff field theory. The (infinite set of) equations are found to be gauge invariant and describe all the massive fields at the interacting level. The exact equations are quadratic in the fields. If, for instance, one solves for the massive modes in terms of massless modes one obtains low energy equations which are non polynomial in massless fields. Thus for the graviton Einstein's equation is obtained.

The free gauge invariant equations for open strings were first obtained by this method in [23]. They describe massless particles, and after dimensional reduction with mass, massive particles. The problem of writing down gauge invariant (free) equations for massless/massive tensor fields with arbitrary symmetry is thus solved. [2] There is another generalization that is also possible: One can do all this in a background curved space time. The requirement is that not only should the equations be gauge invariant but also generally covariant. It turns out that this can also be done - but only *after* dimensional reduction with mass.

The gauge transformations have a particularly simple form in this method. The loop variable is parametrized by $k_\mu(t)$ a generalized momentum coordinate:

$$k_\mu(t) = k_{0\mu} + \frac{k_{1\mu}}{t} + \frac{k_{2\mu}}{t^2} + ... \frac{k_{n\mu}}{t^n} + ...$$

Here $k_{0\mu}$ is the usual space time momentum. The gauge transformations have the simple rescaling form

$$k_\mu(t) \to k_\mu(t)\lambda(t) \tag{1.0.1}$$

[1] There is a technicality here: the EOM is proportional to the beta function, the proportionality factor being the Zamolodchikov metric [14, 15, 13, 18, 17].

[2] For symmetric tensors it was solved many years back in [24, 25].

where $\lambda(t)$ contains the gauge parameters

$$\lambda(t) = \lambda_0 + \frac{\lambda_1}{t} + ... + \frac{\lambda_n}{t^n} + ... \tag{1.0.2}$$

This suggests that gauge transformation in string theory has a geometric space time interpretations as some kind of scale transformations. In [23] it was speculated that the underlying gauge principle in string theory, generalizing the coordinate invariance of Einstein's theory of gravity, is some generalized renormalization group elevated to a gauge symmetry.

An approach to the interacting open string was developed in [26, 16] where the interacting theory was recast as a free theory where the string was, so to speak, thickened to a band with an extra parameter. When the product of two vertex operators had the same value of this parameter it was interpreted as a higher mode vertex operator, and when the value is different, it is interpreted as an interaction term between two fields. The method gave very easily a gauge invariant interacting theory. While this was elegant from the world sheet viewpoint, from the space time perspective, the equations were very complicated.

A more satisfactory interacting theory is obtained by applying ERG transformation [28, 29]. The ERG is a quadratic equation. Thus in this formalism the string field equations are automatically quadratic. This is natural in string theory because the basic interaction is a cubic one - splitting one string into two or joining of two strings to form one. This interacting theory, for open strings, has an interesting property: unlike in string field theory, the interactions do not modify the gauge transformation rule of the fields. The interactions are written in terms of gauge invariant "field strengths". In this sense even though the theory is interacting, it looks Abelian - when the gauge group is $U(1)$. If Chan-Paton factors are added to make the group non Abelian, then the gauge transformation rule is indeed modified. In contrast BRST string field theory [30, 31, 32] looks non Abelian even when the gauge group is U(1) although there are hints that field redefinitions can modify this feature [43] .

The story gets a little more involved for closed strings [33, 34, 35]. This is due to the presence of the massless graviton. At the free level things work out as they should - one obtains linearized equations for the graviton with the usual "Abelian" gauge transformation:

$$\delta h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}$$

For the interactions, if one applies the procedure followed for open strings, one finds that everything works out correctly for a *massive* graviton! The gauge invariant "field strength" can be written down and, in terms of them, interactions also. However this field strength is not gauge invariant if the graviton is massless. It turns out that the resolution of this problem is to modify the gauge transformation law to include a shift of the coordinate X^μ. This relates the gauge transformation to coordinate transformations (as it should in general relativity (GR)) and modifies the gauge transformation rule for the metric fluctuation $h_{\mu\nu}$:

$$\delta h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)} + \epsilon^\lambda h_{\mu\nu,\lambda} + \epsilon^\lambda{}_{,\mu} h_{\lambda\nu} + \epsilon^\lambda{}_{,\nu} h_{\mu\lambda} \tag{1.0.3}$$

The tensorial rotations are the result of including $\delta X^\mu = -\epsilon^\mu$ in the gauge transformation. (Note: $\epsilon_\mu \equiv \eta_{\mu\nu}\epsilon^\nu$.) Thus for closed strings, interactions do indeed modify the gauge transformations and give them a non-Abelian structure.

One can then introduce a background metric and make the theory invariant under background coordinate invariance i.e. a symmetry where not only the physical metric but also the background metric transforms. This is analogous to what happens in the background field formalism for non Abelian gauge theories. This requires that all other terms in the world sheet action be modified so that the theory is coordinate invariant.

At this point there are two options: If the background metric is unrelated to the physical metric, then we have to ensure that somehow the action is further modified so that there is no net dependence on the background metric. In order to achieve this some field redefinitions of the massive modes have to be performed. However if we let the background metric be the same as the physical metric then the theory is correct as it stands.

Once this covariantization is done, we have to study again whether there is a clash between gauge invariance and general coordinate invariance. It turns out that there is a clash for the higher spin fields if we

follow the naive procedure. However, when the higher spin fields are massive, it is possible to get around this. There is a systematic procedure for modifying the equations of motion by adding higher derivative terms involving the Stuckelberg fields coupled to the curvature tensor such that gauge symmetry is preserved. These terms are not there in flat space. Also since the Stuckelberg fields can be set to zero by a gauge transformation, the physical fields continue to have two derivative propagation equations, even in curved space time.

In summary, this procedure gives gauge invariant and general coordinate equations for all the massive modes of the string. This formalism is thus "background independent". Another approach to background independent formalism for string theory is described in [36] and developed further in [37, 38, 39].

There are some open questions as well:

Whether these equations are physically equivalent to those of BRST string field theory has not been checked or investigated in detail. However there is reason to believe that they are equivalent. Being gauge invariant one can always fix gauge and obtain world sheet theory in the old covariant formalism, which is the Polyakov action in conformal gauge without any ghosts. Also imposing conformal (Weyl) invariance at the free level is equivalent to the physical state Virasoro constraints. So the theory reduces exactly to the old covariant formalism and the interacting beta function constraints should be equivalent to the on shell S-matrix, by the same arguments that worked for the tachyon [40].

A related issue that needs to be resolved is about the world sheet action for the extra coordinate. In obtaining the free equations of motion for the string fields, only the coincident two point function of this field is needed. However in calculating the S matrix, if one wants to reproduce the S matrix of the old covariant formalism, this coordinate should not contribute at all. This requires that the Green function vanish everywhere except at coincident points. This is a little unusual - it corresponds to an infinitely massive world sheet field. This needs to be understood better.

This paper is a self contained review of this approach and is organized as follows: Section 2 describes the beta function approach and explains the problems involved in obtaining gauge invariant equations. Section 3 describes the basic loop variable and explains how one obtains the free gauge invariant equations for the open string. Section 4 describes dimensional reduction. Section 5 reviews the ERG and applies it to open strings in flat space. Section 6 gives a prescription for a consistent map from loop variables to space time fields in curved space time in such a way that gauge invariance is preserved. Section 7 describes closed strings in arbitrary backgrounds and gives the gauge invariant formulation. Section 8 describes the connection with old covariant formalism for string theory. Section 9 contains conclusions.

2 Free Equations for Open Strings Fields

In this section we explain the RG method with some examples and explain the issues. We start with the Polyakov action modified by boundary terms describing an open string background.

$$S = \frac{1}{4\pi\alpha'}\int_\Gamma d^2\sigma\{\partial^\alpha X^\mu \partial_\alpha X_\mu\} + \int_{\partial\Gamma} dx \sum_i g^i V_i(x) \tag{2.0.1}$$

Here Γ is the UHP plane (or the unit disc) and $\partial\Gamma$ is the real axis (or the unit circle). The two point function for points on the real axis is obeying Neumann boundary conditions is

$$\langle X(z)X(w)\rangle = 2\alpha' ln(z-w)$$

This will be regularized when $z = w$ to

$$\langle X^\mu(z)X^\nu(z)\rangle = -2\eta^{\mu\nu}\alpha'(ln(a) + \sigma(z))$$

For the open string the vertex operators are all along the real axis where $z = \bar{z}$. So the chiral field $X(z)$ has all the oscillators for the open string and on the real axis the vertex operators corresponding to derivatives of $X(z)$. The cutoff is taken to be ae^σ where σ is the Liouville mode. This simple regularization prescription

is all one needs for the free theory. When we write down the full ERG a more well defined regularization will be required. Even in that case, the exact form of the regulator is immaterial. Different choices correspond to different schemes and the idea of universality is that the physics of the continuum does not depend on these details. In string theory this translates to the statement that the S-matrix will not be affected. Thus different world sheet regularization schemes must correspond to field redefinition of the space-time fields.

2.1 Vertex Operators and β functions

Let us consider some of the simplest vertex operators in turn.

2.1.1 Tachyon

The vertex operator for the tachyon is just $e^{ik_0.X(z)}$. Thus we add to the action

$$\Delta S = \int dk_0\ \phi(k_0) \int_{\partial\Gamma} dz\ \frac{1}{ae^{\sigma}} e^{ik_0.X(z)}$$

We can write the vertex operator as a normal ordered operator:

$$e^{ik_0.X(z)} = e^{-\frac{k_0^2}{2}\langle X(z)X(z)\rangle} : e^{ik_0.X} := e^{\alpha' k_0^2 (ln(a)+\sigma)} : e^{ik_0.X} :$$

Thus

$$\Delta S = \int dk_0\ \phi(k_0) \int_{\partial\Gamma} dz\ \frac{1}{ae^{\sigma}} e^{\alpha' k_0^2 (ln(a)+\sigma)} : e^{ik_0.X} :$$

Requiring either, $\frac{d}{dln(a)}\Delta S = 0$ (vanishing β function) or more precisely $(\frac{d}{d\sigma}\Delta S)|_{\sigma=0} = 0$ (independence from Liouville mode i.e. Weyl invariance) gives the condition:

$$(\alpha' k_0^2 - 1)\phi(k_0) = 0 \tag{2.1.2}$$

Note that this is the same as the Virasoro condition $[L_0, V(z)] = 0$.

2.1.2 Vector

The question of gauge invariance arises for the vector. The vertex operator is $k_{1\mu}\partial_z X^{\mu} e^{ik_0.X(z)}$. One can think of $k_{1\mu}$ as the polarization of the gauge field. But it is more useful to proceed as follows: In the world sheet action we add the background term

$$\int_{\partial\Gamma} dz A_{\mu}(X(z))\partial_z X^{\mu}(z) = \int dk_0 \int dz A_{\mu}(k_0) e^{ik_0.X(z)} \partial_z X^{\mu}(z)$$

It is convenient to write A_{μ} as a moment:

$$A_{\mu}(k_0) = \int [\prod_{\nu=0}^{D} dk_{1\nu}] k_{1\mu} \Psi[k_{0\nu}, k_{1\nu}, A_{\nu}(k_0)] \tag{2.1.3}$$

and think of $k_{1\mu}$ as a generalized momentum, dual to A_{μ}. The "wave function" Ψ has the information about the momentum dependence of $A_{\mu}(k_0)$. It is also convenient to write the vertex operator as $e^{ik_0.X(z)+ik_1.\partial_z X(z)}$ remembering to keep only terms linear in k_1. This can be written in terms of a normal ordered vertex operator with the Liouville mode dependence being made explicit.

$$e^{ik_0.X(z)+ik_1.\partial_z X(z)} = e^{-\frac{1}{2}(k_0.k_0\langle X(z)X(z)\rangle + 2k_0.k_1\langle X(z)\partial_z X(z)\rangle + k_1.k_1\langle \partial_z X(z)\partial_z X(z)\rangle)} : e^{ik_0.X(z)+ik_1.\partial_z X(z)} :$$

In our case we need keep only terms linear in k_1. This gives

$$e^{-\frac{1}{2}k_0.k_0\langle X(z)X(z)\rangle}[i : k_{1\mu}\partial_z X^{\mu} e^{ik_0.X(z)} : -k_0.k_1\langle X(z)\partial_z X(z)\rangle e^{ik_0.X(z)}]$$

$$= e^{-\frac{1}{2}k_0.k_0\sigma(z)}[i : k_{1\mu}\partial_z X^\mu e^{ik_0.X(z)} : -k_0.k_1\frac{1}{2}\partial_z\sigma(z)e^{ik_0.X(z)}]$$

The coefficient of σ gives the L_0 condition $k_0^2 = 0$ and the coefficient of $\partial_z\sigma$ gives the condition $[L_1, V(z)] = 0$. However it is possible to combine the two as follows:

$$\frac{\delta}{\delta\sigma(z)}\int dz\ e^{-\frac{1}{2}k_0.k_0\sigma(z)}[i : k_{1\mu}\partial_z X^\mu e^{ik_0.X(z)} : -k_0.k_1\frac{1}{2}\partial_z\sigma(z)e^{ik_0.X(z)}] = 0$$

$$= [k_0.k_0 ik_{1\mu} - k_0.k_1 ik_{0\mu}] : \partial_z X^\mu e^{ik_0.X(z)} := 0 \tag{2.1.4}$$

We have integrated by parts the z derivative. This is equivalent to the freedom of adding total derivatives to the action. The net result is a gauge invariant equation: If we replace the polarization $k_{1\mu}$ by a longitudinal $\lambda_1 k_{0\mu}$ the equation is invariant.

$$k_{1\mu} \to k_{1\mu} + \lambda_1 k_{0\mu} \tag{2.1.5}$$

If we replace $k_{1\mu}$ by A_μ we get Maxwell's equation with gauge invariance

$$A_\mu(k_0) \to A_\mu(k_0) + k_{0\mu}\Lambda(k_0)$$

where Λ is a gauge parameter which can be related to λ_1 if the wave function is expanded to include the gauge parameters also. Thus let $\Psi[k_{0\nu}, k_{1\nu}, A_\nu(k_0), \lambda_1, \Lambda(k_0)]$ be the wave function satisfying additionally

$$\int [dk_{1\nu} d\lambda_1]\ \lambda_1 \Psi[k_{0\nu}, k_{1\nu}, \lambda_1, A_\nu(k_0), \Lambda(k_0)] = \Lambda(k_0) \tag{2.1.6}$$

Thus in all equations involving the generalized momenta $k_{1\mu}$, the integral over these momenta and Ψ can be understood, and we can convert it to field equations. Schematically we write

$$\langle\lambda_1\rangle = \Lambda(k_0)\ ;\quad \langle ik_{1\mu}\rangle = A_\mu(k_0)$$

Thus

$$\langle k_0.k_0 k_{1\mu} - k_0.k_1 k_{0\mu}\rangle = k_0^2 A_\mu - k_{0\mu}k_0.A$$

which is Maxwell's equation in momentum space.

The gauge invariance of this equation, as explained above corresponds to the freedom to add total derivatives to the world sheet action. This is also the invariance associated with L_{-1} of the Virasoro group.

2.1.3 Spin2

At spin 2 one expects two vertex operators:$\partial_z X^\mu \partial_z X^\nu e^{ik_0.X(z)}$ and $\partial_z^2 X^\mu e^{ik_0.X(z)}$. Thus one can add to the action

$$\Delta S = \int dz\ [-\frac{1}{2}S_{\mu\nu}(X(z))\partial_z X^\mu \partial_z X^\nu + S_{2\mu}(X(z))\partial_z^2 X^\mu]$$

There are two problems:

For spin 2 and higher there should be symmetries associated with L_{-1}, L_{-2} and higher. L_{-1} symmetries correspond to adding total derivatives and is manifest. However there is no such freedom manifest in the world sheet action for L_{-2} or higher. Thus one does not expect the equation obtained by the same method to have any of the required higher gauge symmetries associated with massive modes.

Furthermore if one applies the Weyl invariance constraint on vertex operators such as $\partial^2 X^\mu e^{ik_0.X(z)}$ one obtains Liouville mode dependent terms such as $k_{0\mu}\partial^2\sigma e^{ik_0.X(z)}$. Integrating by parts one obtains a contribution to the equation involving three powers of momenta. This does not lead to an acceptable equation of motion, which should be quadratic in derivatives. This problem gets worse at higher levels.

This the straightforward application of the vanishing β function condition does not lead to a satisfactory equation of motion. In the next section we introduce the loop variable to solve these problems at the free level.

3 Loop Variable for the Open String

3.1 Loop Variable as a Collection of Vertex Operators

We first describe how all the open string vertex operators can be collected in a loop variable. Consider the following loop variable [3] where c refers to a circle about the point z:

$$e^{ik_0 X(z)+i\int_c \; dt \; ak(t)\partial_z X(z+at)} \tag{3.1.1}$$

with

$$k(t) \;=\; k_0 \;+\; \frac{k_1}{t} \;+\; \frac{k_2}{t^2} \;+\; ....$$

a is a short distance cutoff and is useful for keeping track of the dimension of operators. When we Taylor expand the exponential in a power series we get the following terms:

$$e^{ik_0 X(z)+i\int_c \; dt \; ak(t)\partial_z X(z+at)} = e^{ik_0 X(z)}[1+ik_{1\mu}a\partial_z X^\mu - \frac{1}{2}k_{1\mu}k_{1\nu}a^2\partial_z X^\mu \partial_z X^\nu + ik_{2\mu}a^2\partial^2 X^\mu + ...] \tag{3.1.2}$$

We generalize Ψ in the last section to be $\Psi[k_0, k_1, k_2, ..., k_n, ...]$ and define

$$\langle O \rangle \equiv \prod_{n=1,\infty} \int [dk_n d\lambda_n] O \Psi[k_0, k_1, k_2, ..., k_n, ...] \tag{3.1.3}$$

generalizing what was done for the first few levels in Section 2. Thus for instance

$$\begin{aligned} \langle k_{1\mu} k_{1\nu} \rangle &= S_{\mu\nu}(k_0) \\ \langle k_{2\mu} \rangle &= S_{2\mu}(k_0) \end{aligned} \tag{3.1.4}$$

3.2 Gauge Invariant Formulation

3.2.1 Extra variables to parametrize gauge transformations

The loop variable is generalized to [23] (We have set $a = 1$ below)

$$e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X} \tag{3.2.5}$$

$\alpha(t)$ can be thought of as an "einbein" to make the loop variable reparametrization invariant in t. But we will not need this interpretation here. Let us assume the following Laurent expansion:

$$\alpha(t) \;=\; 1 \;+\; \frac{\alpha_1}{t} \;+\; \frac{\alpha_2}{t^2} \;+\; \frac{\alpha_3}{t^3} + ... \tag{3.2.6}$$

We Taylor expand $X(z+t)$ and Laurent expand $k(t), \alpha(t)$. It is useful to define the following combinations:

$$\begin{aligned} Y &= X \;+\; \alpha_1 \partial_z X \;+\; \alpha_2 \partial_z^2 X \;+\; \alpha_3 \frac{\partial_z^3 X}{2} \;+\; ... \;+\; \frac{\alpha_n \partial_z^n X}{(n-1)!} \;+\; ... \\ &= X \;+\; \sum_{n>0} \alpha_n \tilde{Y}_n \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} Y_1 &= \partial_z X \;+\; \alpha_1 \partial_z^2 X \;+\; \alpha_2 \frac{\partial_z^3 X}{2} \;+\; ... \;+\; \frac{\alpha_{n-1}\partial_z^n X}{(n-1)!} \;+\; ... \\ ... & \quad ... \\ Y_m &= \frac{\partial_z^m X}{(m-1)!} \;+\; \sum_{n>m} \frac{\alpha_{n-m}\partial_z^n X}{(n-1)!} \end{aligned} \tag{3.2.8}$$

$$\tag{3.2.9}$$

[3] The integral is actually $\int_c \frac{dt}{2\pi i}$. We suppress the $2\pi i$ in all the equations. We use t (or s) always to parametrize the loop

We also define $\alpha_0 = 1$ then the $>$ signs in the summations above can be replaced by $\geq$.

In terms of these Y_n's we have ($Y_0 \equiv Y$)

$$e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X} = e^{i\sum_n k_n Y_n} \tag{3.2.10}$$

Let us now introduce x_n by the following:

$$\alpha(t) = \sum_{n\geq 0} \alpha_n t^{-n} = e^{\sum_{m\geq 0} t^{-m} x_m} \tag{3.2.11}$$

Thus

$$\begin{aligned}
\alpha_1 &= x_1 \\
\alpha_2 &= \frac{x_1^2}{2} + x_2 \\
\alpha_3 &= \frac{x_1^3}{3!} + x_1 x_2 + x_3
\end{aligned} \tag{3.2.12}$$

They satisfy the property,

$$\frac{\partial \alpha_n}{\partial x_m} = \alpha_{n-m}, \quad n \geq m \tag{3.2.13}$$

Using this we see that

$$Y_n = \frac{\partial Y}{\partial x_n} \tag{3.2.14}$$

and also

$$\frac{\partial^2 Y}{\partial x_m \partial x_n} = \frac{\partial Y}{x_{n+m}} \tag{3.2.15}$$

3.2.2 Generalizing the Liouville mode

We will work with the Y's rather than X and define $\Sigma = \langle Y(z)Y(z)\rangle$. (This is equal to the Liouville mode σ in coordinates where $\alpha(s) = 1$.) We then impose $\frac{\delta}{\delta\Sigma} = 0$ on the normal ordered vertex operator.

We have for the coincident two point functions:

$$\begin{aligned}
\langle Y\ Y\rangle &= \Sigma \\
\langle Y_n\ Y\rangle &= \frac{1}{2}\frac{\partial \Sigma}{\partial x_n} \\
\langle Y_n\ Y_m\rangle &= \frac{1}{2}\left(\frac{\partial^2 \Sigma}{\partial x_n \partial x_m} - \frac{\partial \Sigma}{\partial x_{n+m}}\right)
\end{aligned} \tag{3.2.16}$$

Using this the normal ordering gives the following Σ dependence:

$$\begin{aligned}
e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X} &= e^{i\sum_n k_n Y_n} \\
&= exp\{k_0^2 \Sigma + \sum_{n>0} k_n.k_0 \frac{\partial \Sigma}{\partial x_n} + \\
&\quad \sum_{n,m>0} k_n.k_m \frac{1}{2}\left(\frac{\partial^2 \Sigma}{\partial x_n \partial x_m} - \frac{\partial \Sigma}{\partial x_{n+m}}\right)\} \\
&\quad : e^{i\sum_n k_n Y_n} :
\end{aligned} \tag{3.2.17}$$

3.2.3 Two derivative EOM

The important thing to note is that there are never more than two derivatives acting on Σ and in fact only one when it multiplies $k_n.k_0$. Thus on integrating by parts we never get more than two powers of k_0.

For instance:

$$\frac{\delta}{\delta\Sigma}[k_n.k_m\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_n\partial x_m}-\frac{\partial\Sigma}{\partial x_{n+m}})]:e^{ik_0.Y}:=:(\frac{1}{2}ik_{0\mu}ik_{0\nu}Y_n^\mu Y_m^\nu+ik_{0\mu}Y_{n+m}^\mu)e^{ik_0.Y}:$$

This solves the problem of getting an EOM that has only two derivatives.

3.2.4 Gauge transformations

Actually introduction of $\alpha(t)$ also solves the problem of gauge invariance. For, in order to be allowed to integrate by parts one must integrate over the x_n. Thus we effectively integrate over $\alpha(n)$: $\int\mathcal{D}\alpha(t)\equiv\prod_n dx_n$. But in that case we can always multiply $\alpha(t)$ by $\lambda(t)\equiv e^{\sum_{n\geq 0}y_n t^{-n}}$. This just translates $x_n\rightarrow x_n+y_n$ under which the measure is invariant. But this is the same as (1.0.1)

$$k_\mu(t)\rightarrow\lambda(t)k_\mu(t)$$

This should be a symmetry. We will show that this is in fact a gauge symmetry of the theory, that includes and generalizes the $U(1)$ gauge symmetry of electromagnetism described in (2.1.5).

Thus we expand $\lambda(t)$ in inverse powers of t, with $\lambda_0=1$.

$$\lambda(t)=\sum_n\lambda_n t^{-n}$$

Then we can write (1.0.1) as

$$k_n\rightarrow\sum_{m=0}^{n}\lambda_m k_{n-m} \tag{3.2.18}$$

In order to interpret these equations in terms of space-time fields we need a generalization of ((3.1.3)). They have to be extended to include λ as in (2.1.6). Thus we assume that the string wave-functional is also a functional of $\lambda(t)$: $\Psi[k_0,k_1,...,k_n,...,\lambda_1,\lambda_2,...,\lambda_n,...]$

Thus we let terms involving one λ to be equal to gauge parameters:

$$\begin{aligned}\langle\lambda_1\rangle &= \Lambda_1(k_0)\\ \langle\lambda_1 k_{1\mu}\rangle &= \Lambda_{11\mu}(k_0)\\ \langle\lambda_2\rangle &= \Lambda_2(k_0)\end{aligned} \tag{3.2.19}$$

The gauge transformations ((1.0.1)) thus become, after mapping to space time fields by evaluating $\langle..\rangle$:

$$\begin{aligned}A_\mu(k_0) &\rightarrow A_\mu(k_0)+k_{0\mu}\Lambda_1(k_0)\\ S_{2\mu}(k_0) &\rightarrow S_{2\mu}(k_0)+k_{0\mu}\Lambda_2(k_0)+\Lambda_{11\mu}(k_0)\\ S_{11\mu\nu}(k_0) &\rightarrow S_{11\mu\nu}(k_0)+k_{(\mu 0}\Lambda_{11\nu)}\end{aligned} \tag{3.2.20}$$

3.2.5 Explanation of gauge invariance of EOM and tracelessness of gauge parameters

The mechanism for gauge invariance can be understood as follows: As we have seen, a gauge transformation, which is a translation of x_n, changes the normal ordered loop variable by a total derivative in x_n which doesn't affect the equation of motion. More precisely the gauge variation of the loop variable is a term of the form $\frac{d}{dx_n}[A(\Sigma)B]$, where B doesn't depend on Σ. The coefficient of $\delta\Sigma$ is obtained as

$$\int\quad\delta(\frac{d}{dx_n}[A(\Sigma)B])=\int\quad(\frac{d}{dx_n}(\frac{\delta A}{\delta\Sigma}\delta\Sigma)B+\frac{\delta A}{\delta\Sigma}\delta\Sigma\frac{dB}{dx_n})$$

$$= \int \quad [-\frac{\delta A}{\delta \Sigma}\frac{dB}{dx_n} + \frac{\delta A}{\delta \Sigma}\frac{dB}{dx_n}]\delta\Sigma = 0$$

Here we have used an integration by parts.

When we actually do the calculation, the above argument does not quite work. This is because we have simplified some of the terms using (3.2.15). Let us look at the terms up to level 3 - the pattern easily generalizes. The loop variable including Σ dependence is:

$$\begin{aligned} e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X} &= e^{i\sum_n k_n Y_n} \\ &= exp\{k_0^2\Sigma + k_1.k_0\frac{\partial\Sigma}{\partial x_1} + k_1.k_1\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial\Sigma}{\partial x_2}) + k_2.k_0\frac{\partial\Sigma}{\partial x_2} + \\ &\quad k_1.k_2(\frac{\partial^2\Sigma}{\partial x_1\partial x_2} - \frac{\partial\Sigma}{\partial x_3}) + k_3.k_0\frac{\partial\Sigma}{\partial x_3}\} \\ &\quad : e^{i(k_0.Y + k_1.Y_1 + k_2 Y_2 + k_3.Y_3)} : \end{aligned} \tag{3.2.21}$$

Let us extract the terms involving λ_1 after performing a gauge transformation. One finds in the exponent:

$$\lambda_1[k_0^2\frac{\partial\Sigma}{\partial x_1} + k_1.k_0(\frac{\partial^2\Sigma}{\partial x_1^2} - \frac{\partial\Sigma}{\partial x_2}) + k_1.k_0\frac{\partial\Sigma}{\partial x_2}]$$

$$+\lambda_1[(k_1.k_1 + k_0.k_2)(\frac{\partial^2\Sigma}{\partial x_1\partial x_2} - \frac{\partial\Sigma}{\partial x_3}) + k_2.k_0\frac{\partial\Sigma}{\partial x_3}]$$

$$= \lambda_1\frac{\partial}{\partial x_1}[k_0^2\Sigma + k_1.k_0\frac{\partial\Sigma}{\partial x_1} + k_2.k_0\frac{\partial\Sigma}{\partial x_2}] + \lambda_1 k_1.k_1(\frac{\partial^2\Sigma}{\partial x_1\partial x_2} - \frac{\partial\Sigma}{\partial x_3})$$

If we set $\lambda_1 k_1.k_1 = 0$ then the change in the loop variable is a total derivative in x_1. This will give a gauge invariant contribution to the EOM by the arguments given in the previous paragraph. Since it is in the exponent, the condition $\lambda_1 k_1.k_1 = 0$ actually means $\lambda_1 k_1.k_1(...) = 0$ where the three dots indicate any other combinations of loop momenta k_n. For the gauge parameters, this is a tracelessness condition, which is known to be required in higher spin theories. At higher levels this becomes $\lambda_n k_m.k_p(....) = 0$ for $m, p > 0$.

3.2.6 Spin 2 EOM

Let us apply $\frac{\delta}{\delta\Sigma}|_{\Sigma=0} = 0$ to (3.2.21). One gets very easily:

$$[ik_0^2 k_2.Y_2 - ik_1.k_0 k_1.Y_2 - ik_2.k_0 k_0.Y_2 + ik_1.k_1 k_0.Y_2]e^{ik_0 Y}$$

$$+[k_1.k_0 k_0.Y_1 k_1.Y_1 + \frac{1}{2}k_1.k_1(k_0.Y_1)^2 - \frac{1}{2}k_0^2(k_1.Y_1)^2]e^{ik_0 Y} = 0 \tag{3.2.22}$$

We have separated by square brackets the two vertex operators at level 2. It is very easy to check that the equations are invariant under

$$k_1 \to k_1 + \lambda_1 k_0 \quad ; \quad k_2 \to k_2 + \lambda_1 k_1 + \lambda_2 k_0 \tag{3.2.23}$$

Converting to space time fields (using $\langle ...\rangle$ (3.1.3)) one obtains:

$$\Box S_{2\mu} - \partial^\nu S_{\nu\mu} - \partial_\mu\partial^\nu S_{2\nu} + \partial_\mu S_\nu^\nu = 0 \tag{3.2.24}$$

$$\frac{1}{2}\Box S_{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\nu S_\rho^\rho - \partial_\nu\partial^\rho S_{\rho\mu} = 0 \tag{3.2.25}$$

The gauge transformations are

$$\begin{aligned} S_{2\mu} &\to S_{2\mu} + k_{0\mu}\Lambda_2 + \Lambda_{11\mu} \\ S_{\mu\nu} &\to S_{\mu\nu} + k_{0(\mu}\Lambda_{11\nu)} \end{aligned} \tag{3.2.26}$$

3.2.7 Spin 3 EOM

By the same procedure one finds the EOM for spin 3:

$$
\begin{aligned}
&[-ik_0^2\frac{(k_1.Y_1)^3}{3!}+ik_1.k_0k_0.Y_1\frac{(k_1.Y_1)^2}{2!}-\frac{i}{2}k_1.k_1(k_0.Y_1)^2(k_1.Y_1)]+\\
&[ik_0^2k_3.Y_3-ik_1.k_0k_2.Y_3-ik_2.k_0k_1.Y_3+ik_1.k_1k_1.Y_3-ik_3.k_0k_0.Y_3+2ik_2.k_1k_0.Y_3]+\\
&[-k_0^2k_1.Y_1k_2.Y_2+k_1.k_0k_0.Y_1k_2.Y_2+k_1.k_0k_1.Y_2k_1.Y_1+k_2.k_0k_0.Y_2k_1.Y_1\\
&-k_1.k_1k_0.Y_2k_1.Y_1-k_1.k_1k_0.Y_1k_1.Y_2-k_2.k_1k_0.Y_2k_0.Y_1]=0
\end{aligned}
\tag{3.2.27}
$$

We have collected in square brackets, vertex operators of a given type. Thus these are three separate equations. They are invariant under (3.2.23) along with

$$k_3 \rightarrow k_3+\lambda_1 k_2+\lambda_2 k_1+\lambda_3 k_0$$

Space time fields for Spin 3 are defined later.

4 Dimensional Reduction

The equations obtained above are gauge invariant and describe massless fields. The second equation for the spin 2 level is in fact the free equation for a massless spin 2 tensor such as the graviton. However we are trying to describe the massive spin 2 and the field $S_{2\mu}$ is required for this. We do not have a mass term in any of the equations. This is because what we have done so far is not strictly a beta function calculation. We have not included the σ dependence associated with the engineering (classical) dimensions of the vertex operators - only the anomalous dimension. Including the engineering dimension is difficult to do because we are working with Σ, which is a linear combination of σ and its derivatives.

Now in the BRST formalism it was shown that the extra auxiliary and Stueckelberg fields necessary for a gauge invariant description of string theory are obtained from the oscillators of a bosonized ghost coordinate, minus the first oscillator mode. Motivated by this let us assume that one of the coordinates Y^D is this extra coordinate. Let us call it θ and the corresponding momentum $q(t)$. We then perform a Kaluza-Klein dimensional reduction and let q_0, the internal momentum be the mass of the field. To match with string theory q_0^2 will have to be set equal to the engineering dimension of the vertex operator. [4]So (in some appropriate units) we can set $q_0^2=1$. Thus the engineering dimension arises as the anomalous dimension associated with the extra dimension.

In the BRST formalism some field redefinitions have to be performed to get the fields into a form that is obtainable by dimensional reduction of a massless higher dimensional theory. This redefinition works only in the critical dimension and with the correct string spectrum. Here we automatically get it in that form directly. Furthermore gauge invariance is not affected by dimensional reduction, so q_0^2 can have any value. We can choose it to match the string spectrum. But we will keep it as q_0^2 till we are forced to some specific value. There is also no critical dimension so far. However it was shown in [40] that if one wants the gauge transformations and constraints to be in the standard form of string theory, field redefinitions need to be done, and can be done only in the critical dimension and with the right spectrum.

With these comments out of the way, let us proceed with the dimensional reduction.

4.1 Dimensional Reduction of the free EOM: Spin 2, Spin 3

We first give explicitly the details regarding dimensional reduction of the Spin 2 and Spin 3 system described above.

[4] This means that it is not a simple S^1 compactification, which would have required q_0 to be an integer. We need q_0^2 to be an integer to match the string spectrum.

4.1.1 Spin 2

The *naive* field content is

$$\begin{aligned}
\langle k_{1\mu}k_{1\nu}\rangle &\equiv S_{11,\mu\nu} \qquad (4.1.1)\\
\langle k_{1\mu}q_1\rangle &\equiv S_{11,\mu} \qquad (4.1.2)\\
\langle q_1q_1\rangle &\equiv S_{11} \qquad (4.1.3)\\
\langle k_{2\mu}\rangle &\equiv S_{2,\mu} \qquad (4.1.4)\\
\langle q_2\rangle &\equiv S_2 \qquad (4.1.5)
\end{aligned}$$

The gauge parameters are then

$$\begin{aligned}
\langle \lambda_2\rangle &\equiv \Lambda_2 \qquad (4.1.6)\\
\langle \lambda_1 k_{1\mu}\rangle &\equiv \Lambda_{11,\mu} \qquad (4.1.7)\\
\langle \lambda_1 q_1\rangle &\equiv \Lambda_{11} \qquad (4.1.8)\\
& \qquad (4.1.9)
\end{aligned}$$

The field transformations are

$$\begin{aligned}
S_{2,\mu} &\rightarrow S_{2\mu} + k_{0\mu}\Lambda_2 + \Lambda_{11\mu} \qquad (4.1.10)\\
S_2 &\rightarrow S_2 + q_0\Lambda_2 + \Lambda_{11} \qquad (4.1.11)\\
S_{11,\mu\nu} &\rightarrow S_{11,\mu\nu} + k_{0(\mu}\Lambda_{11\nu)} \qquad (4.1.12)\\
S_{11,\mu} &\rightarrow S_{11,\mu} + k_{0\mu}\Lambda_{11} + q_0\Lambda_{11\mu} \qquad (4.1.13)\\
S_{11} &\rightarrow S_{11} + 2q_0\Lambda_{11} \qquad (4.1.14)\\
& \qquad (4.1.15)
\end{aligned}$$

Thus $S_{11,\mu\nu}, S_{11,\mu}, S_{11}$ form a gauge invariant massive spin 2 and $S_{2\mu}, S_2$ form a massive spin 1. One can gauge away $S_{11,\mu}, S_{11}, S_2$ using $\Lambda_{11\mu}, \Lambda_{11}, \Lambda_2$ and get a gauge fixed covariant description of a massive spin 2 and massive spin 1. However this is not the correct field content for open strings at the first massive level.

Comparison with String Spectrum:

For a string in D dimensions, the physical states (defined by light cone oscillators $\alpha^i_{-2}, \alpha^i_{-1}\alpha^j_{-1}$) are $O(D-2)$ tensors given by: $\square$(4), $\square\square$(10)

They are combined in the $O(D-1)$ symmetric traceless tensor: $\square\square$(15-1=14)

These are the transverse components of a massive $O(D)$ tensor (for which $O(D-1)$ is the little group). Although $D=26$ for the bosonic string we use a smaller number for D, say $D=6$ to specify the size of the rep. Thus the numbers in brackets are the dimensions of the reps for $D=6$. For a covariant description we keep the trace. For a gauge invariant description an additional vector and a scalar are also needed as we have seen above.

4.1.2 Q-rules for level 2

The resolution is to use the identifications (called Q-rules):

$$\begin{aligned}
q_1q_1 &= q_2q_0 \qquad (4.1.16)\\
q_1k_{1\mu} &= q_0k_{2\mu} \qquad (4.1.17)\\
\lambda_1q_1 &= \lambda_2q_0 \qquad (4.1.18)
\end{aligned}$$

These can be used to get rid of q_1 from the equations. This is motivated by the BRST prescription [30] of eliminating the first oscillator in the bosonized ghost field. These identifications are made by requiring that

the LHS and RHS transform the same way under gauge transformation. This ensures that gauge invariance is maintained during the truncation of the field content. Then $S_{2\mu}$ which becomes identified with $S_{11\mu}$ up to a factor of $q_0 \neq 0$ gets gauged away when we gauge away $S_{11\mu 5}$. Also S_{11} and S_2 get identified and can be gauged away. Thus we are left with just $S_{11\mu\nu}$ - which gives the field content for a covariant description of a massive symmetric tensor and reproduces the open string spectrum.

One important point to note is that these Q-rules are consistent with the idea of a higher dimensional origin. Namely, if we let $\mu = 5$ in the second equation, we do get the first equation. For higher levels these q rules are quite non trivial but, very surprisingly, they continue to be consistent with dimensional reduction.

Massive spin 2 equation:

We give the resulting equation for massive spin 2 field.

$$\frac{1}{2}(k_0^2+q_0^2)k_{1\mu}k_{1\nu} - \frac{1}{2}k_0.k_1 k_{1(\mu}k_{0\nu)} + \frac{1}{2}k_{0\mu}k_{0\nu}k_1.k_1 - \frac{1}{2}q_0^2 k_{2(\mu}k_{0\nu)} + \frac{1}{2}k_{0\mu}k_{0\nu}q_2 q_0 = 0 \qquad (4.1.19)$$

4.1.3 Spin 3

$$\langle k_{1\mu}k_{1\nu}k_{1\rho}\rangle \equiv S_{111}^{\mu\nu\rho}; \quad \langle k_{1\mu}k_{1\nu}q_1\rangle \equiv S_{111}^{\mu\nu}; \quad \langle k_{1\mu}q_1 q_1\rangle \equiv S_{111}^{\mu}; \quad \langle q_1 q_1 q_1\rangle \equiv S_{111}$$
$$\langle k_{2\mu}k_{1\nu}\rangle \equiv S_{21}^{\mu\nu}; \quad \langle k_{1\mu}q_2\rangle \equiv S_{12}^{\mu}; \quad \langle k_{2\nu}q_1\rangle \equiv S_{21}^{\nu};$$
$$\langle k_3^{\mu}\rangle \equiv S_3^{\mu}; \quad \langle q_3\rangle \equiv S_3$$

The gauge transformations again follow an obvious pattern:

$$\delta S_{111}^{\mu\nu\rho} = \langle \lambda_1 (k_0^{(\mu}k_{1\nu}k_1^{\rho)}\rangle \equiv k_0^{(\mu}\Lambda_{111}^{\nu\rho)}$$

$$\delta S_{111}^{\mu\nu} = \langle \lambda_1 (q_0 k_{1\mu}k_{1\nu} + q_1 k_0^{(\mu}k_1^{\nu)})\rangle \equiv q_0 \Lambda_{111}^{\mu\nu} + k_0^{(\mu}\Lambda_{111}^{\nu)}$$
$$\delta S_{111}^{\mu} = \langle \lambda_1 (2q_1 q_0 k_{1\mu} + q_1^2 k_{0\mu})\rangle \equiv 2q_0 \Lambda_{111}^{\mu} + k_0^{\mu}\Lambda_{111}$$
$$\delta S_{111} = 3\langle \lambda_1 q_1^2 q_0\rangle \equiv 3q_0 \Lambda_{111}$$
$$\delta S_{21}^{\mu\nu} = \langle \lambda_1 (k_{0\nu}k_{2\mu} + k_{1\mu}k_{1\nu}) + \lambda_2 k_{0\mu}k_{1\nu}\rangle \equiv k_0^{\nu}\Lambda_{21}^{\mu} + \Lambda_{111}^{\mu\nu} + k_0^{\mu}\Lambda_{12}^{\nu}$$
$$\delta S_{12}^{\mu} = \langle \lambda_1 (k_{0\mu}q_2 + q_1 k_{1\mu}) + \lambda_2 k_{1\mu}q_0\rangle \equiv k_0^{\mu}\Lambda_{21} + \Lambda_{111}^{\mu} + \Lambda_{12}^{\mu}q_0$$
$$\delta S_{21}^{\mu} = \langle \lambda_1 (k_{2\mu}q_0 + q_1 k_{1\mu}) + \lambda_2 k_{0\mu}q_1\rangle \equiv \Lambda_{21}^{\mu}q_0 + \Lambda_{111}^{\mu} + k_0^{\mu}\Lambda_{12}$$
$$\delta S_3^{\mu} = \langle \lambda_3 k_{0\mu} + \lambda_2 k_{1\mu} + \lambda_1 k_{2\mu}\rangle \equiv k_0^{\mu}\Lambda_3 + \Lambda_{12}^{\mu} + \Lambda_{21}^{\mu}$$

These describe a massive spin 3, spin 2 and spin 1, as obtained by dimensional reduction of a massless theory in one higher dimension.

Once again this is not the open string spectrum.

Comparison with string spectrum:

The physical states (light cone oscillators $\alpha_{-3}^i, \alpha_{-2}^i\alpha_{-1}^j, \alpha_{-1}^i\alpha_{-1}^j\alpha_{-1}^k$) are $\square$(4), $\square\otimes\square$(=$\square\square$(10)$\oplus$ $\begin{array}{c}\square\\\square\end{array}$(6)) and $\square\square\square$(20) for a total of 40 states.

These can be combined into a symmetric traceless 3-tensor and an antisymmetric 2-tensor of $O(D-1)$ (the little group for a massive particle): $\square\square\square$(35-5=30) , $\begin{array}{c}\square\\\square\end{array}$(10)

Fully covariant description of a three tensor requires a traceless three tensor, a vector (which can be taken to be the trace of a traceful three tensor) and a scalar. Clearly a truncation is required. Once again we specify the Q-rules at this level.

4.1.4 Q-rules for level 3

$$\begin{aligned}
\langle q_1 k_{1\mu} k_{1\nu}\rangle &= \frac{1}{2}\langle k_2^{(\mu} k_1^{\nu)} q_0\rangle = \frac{1}{2} S_{21}^{(\mu\nu)} q_0 \\
\langle q_1 q_1 k_{1\mu}\rangle &= \langle k_3^{\mu} q_0^2\rangle = S_3^{\mu} q_0^2 \\
\langle q_1 k_{2\mu}\rangle &= \langle 2k_3^{\mu} q_0 - q_2 k_{1\mu}\rangle = 2S_3^{\mu} q_0 - S_{12}^{\mu} \\
\langle q_1 q_2 q_0\rangle &= \langle q_3 q_0^2\rangle = \langle q_1^3\rangle = S_3 q_0^2 \\
\langle \lambda_1 q_1 k_{1\mu}\rangle &= \langle \frac{1}{2}\lambda_2 k_{1\mu} q_0 + \frac{1}{2}\lambda_1 k_{2\mu} q_0\rangle = \frac{1}{2}(\Lambda_{12}^{\mu} + \Lambda_{21}^{\mu}) q_0 \\
\langle \lambda_2 q_1\rangle &= \langle 2\lambda_3 q_0 - \lambda_1 q_2\rangle = 2\Lambda_3 q_0 - \Lambda_{21} \\
\langle \lambda_1 q_1 q_1\rangle &= \langle \lambda_3 q_0^2\rangle = \Lambda_3 q_0^2
\end{aligned} \tag{4.1.20}$$

This results in some modifications in gauge transformations.

$$\delta(q_2 k_{1\mu}) = (\frac{3}{2}\lambda_2 k_{1\mu} + \frac{1}{2}\lambda_1 k_{2\mu}) q_0 + \lambda_1 q_2 k_{0\mu} \quad , \quad \delta q_3 = 3\lambda_3 q_0 \tag{4.1.21}$$

The resulting truncated set of fields are given below:

$$\begin{aligned}
\langle k_{1\mu} k_{1\nu} k_{1\rho}\rangle &= S_{111\mu\nu\rho} \\
\langle \frac{1}{2} k_{2[\mu} k_{1\nu]}\rangle &= A_{21\mu\nu} \\
\langle \frac{1}{2} k_{2(\mu} k_{1\nu)} q_0 = k_{1\mu} k_{1\nu} q_1\rangle &= S_{21\mu\nu} q_0 \\
\langle k_{3\mu} q_0^2 = k_{1\mu} q_1^2 = \frac{1}{2}(k_{1\mu} q_2 + k_{2\mu} q_1) q_0\rangle &= S_{3\mu} q_0^2 \\
\langle k_{1\mu} q_2\rangle &= S_{12\mu} \\
\langle q_3 q_0^2 = q_2 q_1 q_0 = q_1^3\rangle &= S_3 q_0^2
\end{aligned} \tag{4.1.22}$$

Note that symmetrization and anti-symmetrization are not normalized to 1. Hence the factor of $\frac{1}{2}$ in the definition of the index tensors. Their gauge parameters and transformations are:

$$\begin{aligned}
\langle \lambda_1 q_1 q_1 = \frac{1}{2}(\lambda_2 q_1 + \lambda_1 q_2) q_0 = \lambda_3 q_0^2\rangle &= \Lambda_3 q_0^2 \\
\langle \frac{1}{2}(\lambda_1 q_2 - \lambda_2 q_1)\rangle &= \Lambda_A q_0 \\
\langle \lambda_1 q_1 k_{1\mu} = \frac{1}{2}(\lambda_2 k_{1\mu} + \lambda_1 k_{2\mu}) q_0\rangle &= \frac{1}{2}(\Lambda_{12\mu} + \Lambda_{21\mu}) q_0 = q_0 \Lambda_S \\
\langle \frac{1}{2}(\lambda_2 k_{1\mu} - \lambda_1 k_{2\mu})\rangle &= \frac{1}{2}(\Lambda_{12\mu} - \Lambda_{21\mu}) = \Lambda_A \\
\langle \lambda_1 k_{1\mu} k_{1\nu}\rangle &= \Lambda_{111\mu\nu}
\end{aligned} \tag{4.1.23}$$

$$\begin{aligned}
\delta S_3 &= 3\Lambda_3 q_0 \\
\delta S_{3\mu} &= 2\Lambda_{S\mu} + k_{0\mu}\Lambda_3 \\
\delta(S_{3\mu} q_0 - S_{12\mu}) \equiv \delta S_{A\mu} q_0 &= \Lambda_{A\mu} q_0 + k_{0\mu}\Lambda_{A\mu} q_0 \\
\delta S_{\mu\nu} &= \Lambda_{111\mu\nu} + k_{0(\mu}\Lambda_{S\nu)} \\
\delta A_{\mu\nu} &= k_{0[\mu}\Lambda_{A\nu]} \\
\delta S_{111\mu\nu\rho} &= k_{0(\mu}\Lambda_{111\nu\rho)}
\end{aligned} \tag{4.1.24}$$

The tracelessness condition on the gauge parameter is $\lambda_1 k_1.k_1 + \lambda_1 q_1 q_1 = 0$. Thus

$$\Lambda_{111}{}^{\mu}{}_{\mu} + \Lambda_3 q_0^2 = 0 \tag{4.1.25}$$

This is the field content for a covariant and gauge invariant equation. All fields other than the three index symmetric tensor and the antisymmetric two tensor can be set to zero by a choice of gauge. This thus agrees with the open string spectrum at level 3. The set of constraints, gauge transformation and connection with the old covariant (OC) formalism is described in Section 8.

Massive spin 3 EOM:

We take the massless equation (3.2.36) and perform the dimensional reduction and apply the q-rules given above. As an illustration we give below the equation corresponding to the vertex operator $Y_1^\mu Y_1^\nu Y_1^\rho$. (In our notation, symmetrization, indicated by curved brackets, is not normalized to 1 - just the sum of permutations.)

$$-\frac{1}{3!}(k_0^2+q_0^2)k_{1\mu}k_{1\nu}k_{1\rho} - \frac{1}{3!}(k_1.k_0 k_{1(\mu}k_{1\nu}k_{1\rho)} + k_{2(\mu}k_{1\nu}k_{0\rho)}q_0^2) - \frac{1}{12}(k_1.k_1 k_{1(\mu}k_{0\rho}k_{0\nu)} + q_0^2 k_{3(\mu}k_{0\nu}k_{0\rho)}) = 0 \tag{4.1.26}$$

4.2 Q-rules for higher levels

The procedure for obtaining the massless equations and subsequent naive dimensional reduction is the same at higher levels. The new ingredient that crops up at each level is the consistent truncation of the spectrum to match with open strings. This is done by getting rid of the states that involve q_1. This is done in a manner that preserves gauge invariance by working out the analog of the q-rules at higher levels. Though we have not attempted to find a systematic procedure that works at all levels, we have explicitly worked out the rules for level 4 and 5. It turns out to involve solving a number of sets of *overdetermined* linear equations for some unknown coefficients. It is interesting that they have a consistent solution. Even more interesting is that they can be chosen to be consistent with the idea of dimensional reduction - a requirement that on the face of it seems completely independent. This is also an overdetermined set which turns out to have a solution. We give the results below. It would be interesting to explore this further and find some underlying pattern.

4.2.1 Q-rules for level 4

The basic procedure in obtaining the Q-rules is as follows. Start with the highest spin field at that level, where the Q-rule is uniquely fixed by the symmetry of the indices. This, in turn, also implies a corresponding Q-rule for the gauge parameter. For example, at level 4 we have $q_1 k_{1\mu}k_{1\nu}k_{1\rho}$. The only possibility is to set it equal to $1/3 k_{2(\mu}k_{1\nu}k_{1\rho)}$. The factor 1/3 compensates for the three permutations. Quite generally we choose the sum of the coefficients on the RHS to be 1.

Now consider the gauge transformation of the LHS: $q_1\lambda_1 k_{0(\mu}k_{1\nu}k_{1\rho)} + \lambda_1 q_0 k_{1\mu}k_{1\nu}k_{1\rho}$. Matching the coefficient of $k_{0\rho}$ on both sides gives immediately:

$$\mathcal{Q} : q_1\lambda_1 k_{1\mu}k_{1\nu} \to 1/3(\lambda_2 k_{1\mu}k_{1\nu} + \lambda_1 k_{2\mu}k_{1\nu} + \lambda_1 k_{1\mu}k_{2\nu})$$

This is a general pattern. Once we write down a Q-rule for the fields with n indices, this fixes some Q-rules for gauge parameters with $n-1$ indices. Thus for instance the Q-rule for the two index field [5]:

$$\mathcal{Q} : q_1 k_{2(\mu}k_{1\nu)} \to \frac{A}{2} q_0 k_{3(\mu}k_{1\nu)} + B q_2 k_{1\mu}k_{1\nu} + C k_{2\mu}k_{2\nu}$$

We immediately get constraints on A, B, C matching the gauge parameters on both sides: $A = 6-4C, B = 3C-4$. Also by comparing coefficients of $k_{0\mu}$ we get Q-rules for gauge parameters (with one index, such as $q_1\lambda_2 k_{1\mu}, q_1\lambda_1 k_{2\mu}$) in terms of A, B, C.

[5] Note that the antisymmetric combination, $q_1 k_{2[\mu}k_{1\nu]}$, is uniquely fixed to be $q_0 k_{3[\mu}k_{1\nu]}$

This continues till we have the full set of Q-rules for level 4. The results for the remaining fields are given below:

$$
\begin{aligned}
q_1 k_{2\mu} k_{1\nu} &= \frac{1}{2}\Big(\frac{A}{2} q_0 k_{3(\mu} k_{1\nu)} + B q_2 k_{1\mu} k_{1\nu} + C q_0 k_{2\mu} k_{2\nu} + q_0 k_{3[\mu} k_{1\nu]}\Big) \\
q_1^2 k_{1\mu} k_{1\nu} &= q_0\Big(\frac{A_2}{2} q_0 k_{3(\mu} k_{1\nu)} + B_2 q_2 k_{1\mu} k_{1\nu} + C_2 q_0 k_{2\mu} k_{2\nu}\Big) \\
q_1 k_{3\mu} &= \Big(A_1 q_0 k_{4\mu} + B_1 q_2 k_{2\mu} + C_1 q_3 k_{1\mu}\Big) \\
q_1^2 k_{2\mu} &= q_0\Big(A_3 q_0 k_{4\mu} + B_3 q_2 k_{2\mu} + C_3 q_3 k_{1\mu}\Big) \\
q_1^3 k_{1\mu} &= q_0^2\Big(A_4 q_0 k_{4\mu} + B_4 q_2 k_{2\mu} + C_4 q_3 k_{1\mu}\Big) \\
q_1 q_2 k_{1\mu} &= q_0\Big(A_5 q_0 k_{4\mu} + B_5 q_2 k_{2\mu} + C_5 q_3 k_{1\mu}\Big) \\
q_1 q_3 &= a_1 q_4 q_0 + b_1 q_2^2 \\
q_1^2 q_2 &= a_2 q_4 q_0^2 + b_2 q_0 q_2^2 \\
q_1^4 &= a_3 q_4 q_0^3 + b_3 q_0^2 q_2^2
\end{aligned}
\tag{4.2.27}
$$

The corresponding Q-rules for gauge transformations are:

$$
\begin{aligned}
q_1 \lambda_2 k_{1\nu} &= \frac{1}{2}[(1+\frac{A}{2}) q_0 \lambda_3 k_{1\nu} + (\frac{A}{2}-1) q_0 \lambda_1 k_{3\nu} + C q_0 \lambda_2 k_{2\nu} + B q_2 \lambda_1 k_{1\nu}] \\
q_1 \lambda_1 k_{2\nu} &= \frac{1}{2}[(-1+\frac{A}{2}) q_0 \lambda_3 k_{1\nu} + (\frac{A}{2}+1) q_0 \lambda_1 k_{3\nu} + C q_0 \lambda_2 k_{2\nu} + B q_2 \lambda_1 k_{1\nu}] \\
q_1^2 \lambda_1 k_{1\nu} &= \frac{A_2}{2} q_0^2 (\lambda_3 k_{1\nu} + \lambda_1 k_3^{\nu}) + C_2 q_0^2 \lambda_2 k_{2\nu} + B_2 q_0 q_2 \lambda_1 k_{1\nu} \\
q_1 \lambda_3 &= A_1 q_0 \lambda_4 + B_1 \lambda_2 q_2 + C_1 \lambda_1 q_3 \\
q_1^2 \lambda_2 &= q_0 (A_3 q_0 \lambda_4 + B_3 \lambda_2 q_2 + C_3 \lambda_1 q_3) \\
q_1^3 \lambda_1 &= q_0^2 (A_4 q_0 \lambda_4 + B_4 \lambda_2 q_2 + C_4 \lambda_1 q_3) \\
q_1 q_2 \lambda_1 &= q_0 (A_5 q_0 \lambda_4 + B_5 \lambda_2 q_2 + C_5 \lambda_1 q_3)
\end{aligned}
\tag{4.2.28}
$$

All the parameters turn out to be fixed in terms of two, (which we take to be C, B_2) when we require consistency with gauge transformations.

The general two parameter solution is given below:

$$\{A = 6 - 4C, \quad B = 3C - 4\}$$

$$\{A_1 = \frac{3(C-2)}{2-3C}, \quad B_1 = \frac{6-5C}{2-3C}, \quad C_1 = \frac{2-C}{2-3C}\}$$

$$\{A_2 = \frac{2-4B_2}{3}, \quad C_2 = \frac{1+B_2}{3}\}$$

$$\{A_3 = \frac{3C+2B_2-6}{2-3C}, \quad B_3 = \frac{2(10-2B_2-9C)}{3(2-3C)}, \quad C_3 = \frac{2(2-B_2)}{3(2-3C)}\}$$

$$\{A_4 = \frac{2+6B_2-3C}{2-3C}, \quad B_4 = -\frac{4B_2}{2-3C}, \quad C_4 = -\frac{2B_2}{2-3C}\}$$

$$\{A_5 = -\frac{2B_2}{2-3C}, \quad B_5 = \frac{2-4B_2-3C}{3(2-3C)}, \quad C_5 = \frac{2(2-B_2-3C)}{3(2-3C)}\}$$

$$\{a_1 = \frac{3(2-C)}{2(1+B_2)}, \quad b_1 = \frac{2B_2+3C-4}{2(1+B_2)}\}$$

$$\{a_2 = \frac{1-2B_2}{1+B_2}, \quad b_2 = \frac{3B_2}{1+B_2}\}$$

$$\{a_3 = \frac{3C-5B_2-2}{1+B_2}, \quad b_3 = \frac{3(1+2B_2-C)}{1+B_2}\} \tag{4.2.29}$$

Consistency with Dimensional Reduction

We explain the issue of this consistency with an example. Consider the term $q_1 k_{1\mu} k_{1\nu} k_{1\rho}$. According to the Q-rules this is equal to

$$\mathcal{Q}: q_1 k_{1\mu} k_{1\nu} k_{1\rho} \to \frac{1}{3}(k_{2\mu} k_{1\nu} k_{1\rho} + k_{2\rho} k_{1\mu} k_{1\nu} + k_{2\nu} k_{1\rho} k_{1\mu})$$

Now dimensionally reduce both terms, choosing ρ to be θ. If this dimensional reduction commutes with the Q-rule it should be true that

$$\mathcal{Q}: q_1^2 k_{1\mu} k_{1\nu} = \mathcal{Q}: \frac{1}{3}(q_1 k_{2\mu} k_{1\nu} + q_1 k_{2\nu} k_{1\mu} + q_2 k_{1\mu} k_{1\nu})$$

The two parameter family of Q-rules in fact gives:

$$q_1^2 k_{1\mu} k_{1\nu} = q_0 \Big(\frac{A_2}{2} q_0 k_3^{(\mu} k_1^{\nu)} + B_2 q_2 k_{1\mu} k_{1\nu} + C_2 k_{2\mu} k_{2\nu} \Big)$$

with $A_2 = \frac{2-4B_2}{3}, C_2 = \frac{1+B_2}{3}$. Similarly

$$q_1 k_{2\mu} k_{1\nu} + q_1 k_{2\nu} k_{1\mu} = \Big(\frac{A}{2} q_0 k_3^{(\mu} k_1^{\nu)} + B q_2 k_{1\mu} k_{1\nu} + C k_{2\mu} k_{2\nu} \Big)$$

with $A = 6-4C, B = -4+3C$. Requiring agreement fixes $C = 1+B_2$, thus fixing one parameter. Continuing this process one more step by setting $\nu = \theta$ gives one more constraint and fixes $C = 1, B_2 = 0$. Interestingly, all other constraints for all other terms are satisfied with this choice.

We give the final solution below:

$$\{A = 2, \quad B = -1, \quad C = 1\}$$

$$\{A_1 = 3, \quad B_1 = -1, \quad C_1 = -1\}$$

$$\{A_2 = \frac{2}{3}, \quad B_2 = 0, \quad C_2 = \frac{1}{3}\}$$

$$\{A_3 = 3, \quad B_3 = -\frac{2}{3}, \quad C_3 = -\frac{4}{3}\}$$

$$\{A_4 = 1, \quad B_4 = 0, \quad C_4 = 0\}$$

$$\{A_5 = 0, \quad B_5 = \frac{1}{3}, \quad C_5 = \frac{2}{3}\}$$

$$\{a_1 = \frac{3}{2}, \quad b_1 = -\frac{1}{2}\}$$

$$\{a_2 = 1, \quad b_2 = 0\}$$

$$\{a_3 = 1, \quad b_3 = 0\} \tag{4.2.30}$$

4.2.2 Q-rules for Level 5

Q rules for Level 5 are quite complicated and are given in the Appendix C (C). Once again one obtains a highly overdetermined set of linear equations, which turn out to have a four parameter set of solutions. Requiring consistency with dimensional reduction gives another overdetermined set of equations for these parameters, which again turn out to have a unique solution. It would be interesting to understand the underlying pattern that guarantees the existence of such solutions.

To summarize: using the information presented thus far it is easy to write down gauge invariant *free* equations for all the higher spin modes of string theory. In order to truncate the spectrum consistently one needs the Q-rules which have been given here up to level 5. For higher levels these have to be worked out. The gauge transformation has a simple form - it looks like a scale transformation.

We now need to describe interactions. For this purpose we will generalize the scale (Weyl) invariance of the free theory to the exact renormalization group (ERG) symmetry. This will automatically give us the interactions.

5 Exact Renormalization Group in String Theory

In this section we review some basic facts about the ERG, its connection with the continuum β-function, and how this applies to the world sheet description of string theory.

5.1 Generalities about the RG

We recall the arguments of [19, 20, 21, 22] as applied to string theory. The world sheet action for the bosonic string in general open string backgrounds has the generic form:

$$\begin{aligned} S &= \frac{1}{2}\int_\Gamma d^2\sigma\{\partial^\alpha X^\mu \partial_\alpha X_\mu\} + \int_{\partial\Gamma} dt L_1 \\ L_1 &= \sum_i g^i M_i + \sum_i w^i W_i + \sum_i \mu^i R_i \end{aligned} \tag{5.1.1}$$

μ runs from $0 \ - \ D-1$. D is 26 for the bosonic string. $d^2\sigma$ is the area element in real coordinates and dt the line element. Here Γ denotes the (Euclidean) world sheet. Thus at tree level Γ is a disc (or upper half plane). $\partial\Gamma$ denotes the boundary of Γ. Thus $d^2\sigma = dxdy$ and $dt = dx$ for the upper half plane.

L_1 corresponds to the boundary action corresponding to condensation of open string modes. (For concreteness we restrict ourselves to open string backgrounds, which are boundary terms.) We denote by M_i , W_i, and R_i, marginal, irrelevant and relevant operators respectively. g^i , w^i , μ^i, are the corresponding coupling constants. In string theory, on shell vertex operators are marginal. If k is the *space time* momentum of a string mode with mass m, the length dimension of the corresponding vertex operator increases as k^2, and when $k^2 = m^2$ it is marginal.

The world sheet theory is defined with an ultraviolet cutoff, Λ. Thus the partition function is

$$\int_{|p|<\Lambda} [dX(p)] exp\{-S[X(p), g_i, w_i, \mu_i]\} \tag{5.1.2}$$

It is more convenient to discuss a finite RG "blocking" transformation that takes the cutoff Λ to $\frac{\Lambda}{2}$, rather than making an infinitesimal change. Denote it by $\mathcal{R}$. Thus $\mathcal{R}$ is to be implemented as follows:
1. Perform the integral $\int_{\frac{\Lambda}{2}<|p|<\Lambda}[dX(p)]exp\{-S[X(p)]\}$.
2. Rescale momenta: Let $p' = 2p$. Now the range of p' is again $0 - \Lambda$.
3. Rescale the surviving $X(p)$, $0 < |p| < \frac{\Lambda}{2}$. Let $X(p) = ZX'(p')$. Choose Z so that the kinetic term $p^2X(p)X(-p)$ has the same normalization as before.

There is a subtlety here that needs to be pointed out. The scaling of X in general changes the integration measure $[dX(p)]$. This is related to the trace anomaly in a field theory. In string theory this is related to the β function associated with a closed string mode - the dilaton. We will postpone this point to a later section.

As a result of all the above we get for the partition function:

$$\int_{|p'|<\Lambda} [dX'(p')]exp\{-S[X'(p'), g'_i, w'_i, \mu'_i]\} \tag{5.1.3}$$

which is exactly the same as before except that the coupling constants have different values. Thus effectively

$$\mathcal{R} \ : \ (g, w, \mu) \longrightarrow (g', w', \mu') \tag{5.1.4}$$

defines the discrete renormalization group transformation.

We can then define a recursion relation between the coupling constants:

$$\mathcal{R} \ : \ (g_l, w_l, \mu_l) \longrightarrow (g_{l+1}, w_{l+1}, \mu_{l+1}) \tag{5.1.5}$$

At a fixed point, the couplings satisfy:

$$\mathcal{R} \ : \ (g^*, w^*, \mu^*) \longrightarrow (g^*, w^*, \mu^*) \tag{5.1.6}$$

In string theory there are an infinite number of vertex operators, but to keep the discussion simple we keep one of each type.

The recursion relation thus takes the form

$$\begin{aligned} \mu_{l+1} &= 4\mu_l + N_\mu[\mu_l, g_l, w_l] \\ g_{l+1} &= g_l + N_g[\mu_l, g_l, w_l] \\ w_{l+1} &= \frac{1}{4}w_l + N_w[\mu_l, g_l, w_l] \end{aligned} \tag{5.1.7}$$

where the factor 4 characterizes a dimension-2 relevant operator (eg. a mass term, X^2) and the factor $1/4$ characterizes a dimension-4 irrelevant operator, say of the form $(\partial X \partial X)^2$. N_μ, N_g and N_w correspond to higher order corrections which we take to be small in perturbation theory.

At a fixed point one has

$$\begin{aligned} g_l &= g_{l+1} \\ w_l &= w_{l+1} \\ \mu_l &= \mu_{l+1} \end{aligned} \tag{5.1.8}$$

which means that doing a block transformation does not change anything. This can only be true if there are no dimensionful physical quantities with which to compare the cutoff Λ. Otherwise we would be able to tell the difference between Λ and $\Lambda/2$. Thus the theory has an overall scale - the cutoff, Λ, and no other scale. So correlation lengths are either infinite or zero.

It is important to note that $w^* \neq 0$ in general. In the context of string theory at low (space time) energies massive modes are irrelevant couplings. Thus it is not true in general that the massive modes have zero vev's. Nevertheless we can eliminate the massive modes using their equations of motion and obtain equations of motion for the massless modes - which at low energies correspond to marginal operators. We can understand this in terms of the recursion relations. We follow the discussion of Wilson [20].

Let $0 \leq l \leq L$ be the range of the index l, with μ_0, g_0, w_0 being the parameters of the action at high energies. From eqn. ((5.1.7)) we see that to lowest order $\mu_L \approx 4^L \mu_0$. This blows up rapidly with L. This is the famous "fine-tuning" problem: μ_0 has to be tuned very accurately for μ_L, the low energy effective parameter to have some observed value. Thus it is better to use μ_L as our input. The irrelevant w_0, on the other hand keeps getting smaller and can be used as an input parameter. This way it can be seen easily that

μ_l, w_l, g_l rapidly lose their dependence on w_0: this is the statement of universality. The marginal coupling, g_l is important for all values of l.

Now let us iterate the equations (5.1.7) a number of times. The equations are:

$$\mu_l = 4^{l-L}\mu_L - \sum_{n=l}^{L-1} 4^{l-(n+1)} N_\mu[\mu_n, g_n, w_n] \tag{5.1.9}$$

$$w_l = 4^{-l} w_0 + \sum_{n=0}^{l-1} 4^{n+1-l} N_w[\mu_n, g_n, w_n] \tag{5.1.10}$$

$$g_l = g_{l_0} + \sum_{n=l_0}^{l-1} N_g[\mu_n, g_n, w_n] \tag{5.1.11}$$

We can solve these equation iteratively with the following starting inputs obtained by neglecting the non-linear corrections:

$$\begin{aligned} \mu_l &= 4^{l-L}\mu_L \\ w_l &= 4^{-l} w_0 \\ g_l &= g_{l_0} \end{aligned} \tag{5.1.12}$$

The solution in general has the form

$$g_l = V_g(g_{l_0}, \mu_L, w_0, l, l_0, L)$$

and similarly for μ_l, w_l.

The solution simplifies when $l >> 0$ and $l << L$. Namely the dependence on μ_L and w_0 of g_l is so weak ($O(4^{l-L}$ and 4^{-l}) that we can set $\mu_L = w_0 = 0$ with negligible error. Furthermore the summations can be extended to $+\infty$ for μ_l and $-\infty$ for w_l. The resulting equations have a translational invariance in l and l_0. Thus

$$g_l = V_g[l - l_0, g_{l_0}] \tag{5.1.13}$$

$l - l_0$ is the log of the ratio of the scales and g is dimensionless to begin with. There are no absolute scales in this equation. Furthermore in this region the recursion relation can be approximated by a differential equation, the usual Gell-Mann - Low, Callan-Symanzik β-function involving just the marginal coupling. The solution of this gives us g^*. In string theory this corresponds to EOM for the massless fields, in which the massive fields have been integrated out. Classically, this means solving their EOM. One can also solve for the fixed point value w^*, which is not zero. Thus the massive mode expectation values are not zero. What is nice is that we don't need them to solve for g^*.

If we perform an infinitesimal RG transformation the equations (i.e. N_w, N_g, N_μ) are polynomial in the couplings, while the β functions are non polynomial. This is exactly analogous to the relation between string field theory, which is polynomial in the fields and the low energy EOM, which are not. BRST open string field theory has a cubic action and a quadratic EOM. The ERG is also quadratic in the fields.

There is one more important point: This is the fact that the ERG is obviously written with a finite cutoff. One can perform an infinite number of RG iterations and reach the continuum. All theories on the same RG trajectory describe the same physics. The information about the cutoff scale is contained in the values of the couplings. However, if the couplings are tuned to the values at the fixed point, the value of the cutoff does not matter. It can be kept finite or taken to infinity or to zero. This can be illustrated by just considering the normal ordering corrections of a vertex operator. $\frac{e^{ik.X}}{a} = e^{\frac{(k^2-1)ln(a)}{2}} : e^{ik.X} :$. Thus when $k^2 = 2$, the dependence on the cutoff disappears. At this level of approximation this is the fixed point. Conversely, off shell (read away from a fixed point) one needs to keep a finite cutoff. In string field theory the role of the cutoff is played by the string length, which is finite.

To summarize: the exact RG in this approach gives equation analogous to those of string field theory. It can be taken to be an alternative way of writing down string field theory equations. What we will see in the following is that these equations can be made gauge invariant, as in BRST string field theory. Not only that, the world sheet formalism is manifestly background independent, so we obtain a background independent formalism.

5.2 Exact RG in Position Space

The discussion here follows [19] and [21]. The exact RG [19] originally was written in momentum space. We work in position space because it is more convenient for two dimensional theories. We start with a quantum mechanical version and then generalize in a straightforward way to field theory.

5.2.1 Quantum Mechanics

Consider the Schroedinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial y^2} \tag{5.2.14}$$

for which the Green's function is $\frac{1}{\sqrt{2\pi(t_2-t_1)}}e^{i\frac{(y_2-y_1)^2}{2(t_2-t_1)}}$. Let us change variables :$y = xe^\tau, it = e^{2\tau}$ and $\psi' = e^\tau\psi$. We get

$$\frac{\partial\psi'}{\partial\tau} = \frac{1}{2}\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + x)\psi' \tag{5.2.15}$$

The Green's function is:

$$G(x_2,\tau_2;x_1,0) = \frac{1}{\sqrt{2\pi(1-e^{-2\tau_2})}}e^{-\frac{(x_2-x_1e^{-\tau_2})^2}{2(1-e^{-2\tau_2})}} \tag{5.2.16}$$

Thus as $\tau_2 \to \infty$ it goes over to $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_2^2}$. As $\tau_2 \to 0$ it goes to $\delta(x_1 - x_2)$.

$$\psi(x_2,\tau_2) = \int dx_1 G(x_2,\tau_2;x_1,0)\psi(x_1,0)$$

So

$$\psi(x_2,0) = \psi(x_2)$$

which has all the information about the original function.

$$\psi(x_2,\infty) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_2^2}\int dx_1\psi(x_1)$$

where x_1 is integrated over fully and all information about the original function is lost. Thus consider

$$\frac{\partial}{\partial\tau}\psi(x_2,\tau) = \frac{1}{2}\frac{\partial}{\partial x_2}(\frac{\partial}{\partial x_2} + x_2)\psi(x_2,\tau) \tag{5.2.17}$$

with initial condition $\psi(x,0)$ Thus we can define $Z(\tau) = \int dx_2\psi(x_2,\tau)$, where ψ obeys the above equation, we see that $\frac{d}{d\tau}Z = 0$. Although Z is independent of τ, the integrand changes from being the original function, to a Gaussian times the original function integrated over.

Now take the initial wave function as $e^{\frac{i}{\hbar}S[x]}$ where x denotes the space-time coordinates. Then for $\tau = \infty$ $\psi \approx \int \mathcal{D}x e^{iS[x]}$ is the integrated partition function. At $\tau = 0$ it is the unintegrated $e^{iS[x]}$. τ can parametrize a cutoff so that the high momentum modes get integrated out first.

Following [21], we shall also split the action into a kinetic term and interaction term. Thus we write $\psi = e^{-\frac{1}{2}x^2 f(\tau)+L(x)}$ in the quantum mechanical case discussed above.

By choosing a, b, B suitably ($b = 2af, B = \frac{\dot{f}}{bf}$) in

$$\frac{\partial \psi}{\partial \tau} = B\frac{\partial}{\partial x}(a\frac{\partial}{\partial x} + bx)\psi(x,\tau)$$

we get

$$\frac{\partial L}{\partial \tau} = -\frac{\dot{f}}{2f} + \frac{\dot{f}}{2f^2}[\frac{\partial^2 L}{\partial x^2} + (\frac{\partial L}{\partial x})^2] \tag{5.2.18}$$

Note that if $f = G^{-1}$ (G can be thought of as the propagator) then $\frac{\dot{f}}{f^2} = -\dot{G}$. The first term on the RHS is a field independent constant term, which is normally neglected. The effect of neglecting this term is that the partition function satisfies

$$\frac{dZ}{d\tau} = \frac{\dot{f}}{2f}Z$$

or

$$\frac{d(ln\ Z)}{d\tau} = \frac{\dot{f}}{2f}$$

If we add to L a field independent but τ dependent term, we can get rid of this. This is equivalent to adding a "counterterm" for the cosmological constant. In field theory however, if the background is curved, this term produces a non local metric dependent term which cannot be canceled by a local counterterm. This is the trace anomaly.

In most discussions of the exact RG in flat space this field independent term is neglected.

5.3 Field Theory

We now apply this to a Euclidean field theory.

$$\psi = e^{-\frac{1}{2}\int dz \int dz' X(z)G^{-1}(z,z')X(z') + \int dz L[X(z),X'(z)]} \tag{5.3.19}$$

Here $X'(z) = \partial_z X(z)$. There is a straightforward generalization to the case where higher derivatives $X''(z), X'''(z)$... involving many coordinates. This will be required when we work with loop variables where there are an infinite number of x_n.

Generalizing the quantum mechanical case we apply

$$\int dz \int dz' B(z,z') \frac{\delta}{\delta X(z')}[\frac{\delta}{\delta X(z)} + \int b(z,z'')X(z'')] \tag{5.3.20}$$

to ψ and require that this should be equal to $\frac{\partial \psi}{\partial \tau}$.

$$\frac{\delta}{\delta X(z)}\psi = [-\int du'\ G^{-1}(z,u')X(u') + \frac{\delta}{\delta X(z)}\int du\ L[X(u),X'(u)]]\psi$$

$$\frac{\delta^2}{\delta X(z)\delta X(z')}\psi = [-G^{-1}(z,z') + \frac{\delta^2}{\delta X(z)\delta X(z')}\int du\ L[X(u),X'(u)]]\psi +$$

$$[-\int du'\ G^{-1}(z',u')X(u') + \frac{\delta}{\delta X(z')}\int du'\ L[X(u'),X'(u')]]\times$$

$$[-\int du\ G^{-1}(z,u)X(u) + \frac{\delta}{\delta X(z)}\int du\ L[X(u),X'(u)]]\psi$$

The operator (5.3.20) thus becomes

$$[-G^{-1}(z,z') + \frac{\delta^2}{\delta X(z)\delta X(z')}\int du\ L[X(u),X'(u)]]\psi +$$

$$[-\int du'\ G^{-1}(z',u')X(u') + \frac{\delta}{\delta X(z')}\int du'\ L[X(u'),X'(u')]]\times$$

$$[-\int du\ G^{-1}(z,u)X(u) + \frac{\delta}{\delta X(z)}\int du\ L[X(u),X'(u)]]\psi +$$

$$b(z,z')\psi + \int dz''\ b(z,z'')X(z'')[-\int du'\ G^{-1}(z',u')X(u') + \frac{\delta}{\delta X(z')}\int du'\ L[X(u'),X'(u')]]\psi$$

Now choose $b(z,z') = 2G^{-1}(z,z')$ and $B(z,z') = -\frac{1}{2}\dot{G}(z,z')$. The final equation simplifies to

$$\int du\ \frac{\partial L}{\partial \tau}\psi = \int dz dz'\ \{-\frac{1}{2}\underbrace{\dot{G}(z,z')G^{-1}(z,z')}_{field\ independent} -$$

$$\frac{1}{2}\dot{G}(z,z')[\frac{\delta^2}{\delta X(z)\delta X(z')}\int du\ L[X(u),X'(u)]] + \frac{\delta}{\delta X(z)}\int du\ L[X(u)]\frac{\delta}{\delta X(z')}\int du'\ L[X(u')]\}\psi = 0 \quad (5.3.21)$$

The field independent first term is usually dropped in discussions of the ERG but here we keep it. It has information about the trace anomaly which is proportional to the central charge. It is also supposed to contribute to the dilaton equation. In this subsection we focus on the free EOM which is obtained from the first term.

We now take $\tau \approx ln\ a$. This comes from the scale dependence of the determinant (diagrammatically, the vacuum bubble).

$$\frac{d}{d\tau}\frac{1}{2}Tr\ ln\ G = \frac{1}{2}Tr[\dot{G}G^{-1}]$$

The remaining terms which are the ones usually considered in RG studies, are diagrammatically easy to understand also [21]: the first term in the RHS, which is linear in L, represents contractions of fields at the same point - self contractions within an operator. These can be understood as a pre-factor multiplying normal ordered vertex operators. The second term represents contractions between fields at two different points - between two different operators. For string theory purposes, the first term gives the free equations of motion and the second gives the interactions.

5.4 ERG and Loop Variables: Open Strings

The loop variable represents an infinite collection of all possible vertex operators of the open string. Thus the boundary term in (5.1.1), which is written as $\int dz\ L[X(z),X'(z)...]$ in (5.3.19) is just:

$$\int [dz]\ \mathcal{L}(z) = \int dz\ \mathcal{D}\alpha(t) e^{i\int_c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X}$$

$$= \int \underbrace{[dz dx_1 dx_2 ... dx_n ...]}_{[dz]}\ \ e^{i\sum_n k_n Y_n} = \int [dz]\ \ \mathcal{L}[Y(z,x_n), \frac{\partial Y}{\partial x_1}, \frac{\partial Y}{\partial x_2}, ..., \frac{\partial Y}{\partial x_n}] \quad (5.4.22)$$

From here on the variable z will stand for $(z, x_1, x_2, ..., x_n, ...)$. And z, z', will stand for the sets of variables:

$$(z_A, x_{1A}, x_{2A}, ..., x_{nA}, ...), (z_B, x_{1B}, x_{2B}, ..., x_{nA}, ...)$$

The integrals $\int dz$ in the ERG will be replaced by $\int ... \int dz dx_{1A} dx_{2A} .. dx_{nA} ....$ Variables x_n without a subscript will stand for the integration variable u in the loop variable. Thus we will be allowed to integrate by parts on u, i.e. x_n's. This is responsible for gauge invariance. The free gauge invariant equations obtained in Section 3 will be reproduced. The interaction equation requires a further modification described later below.

5.4.1 Free Equations

$$\int du\ \frac{\delta}{\delta Y^{\mu}(z')}\mathcal{L}(u) = \int du\ \Big\{\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}(u)}\delta(u-z') + \frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_1(u)}\partial_{x_1}\delta(u-z')$$

$$+\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_2(u)}\partial_{x_2}\delta(u-z') + ... + \frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_n(u)}\partial_{x_n}\delta(u-z')\Big\} \tag{5.4.23}$$

We can integrate by parts to get:

$$= \int du\ \Big\{\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}(u)}\delta(u-z') - [\partial_{x_1}\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_1(u)}]\delta(u-z')$$

$$-[\partial_{x_2}\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_2(u)}]\delta(u-z') + ... - [\partial_{x_n}\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_n(u)}]\delta(u-z')\Big\} \tag{5.4.24}$$

This second form is convenient for the interacting term which is a product of first derivatives. For the free case, the first version is better.

$$\frac{\delta^2}{\delta Y^{\mu}(z)\delta Y^{\mu}(z')}\int du\ \mathcal{L}(u) =$$

$$\frac{\delta}{\delta Y^{\mu}(z)}\int du\ \frac{\delta}{\delta Y^{\mu}(z')}\mathcal{L}(u) = \frac{\delta}{\delta Y^{\mu}(z)}\int du\ \Big\{\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}(u)}\delta(u-z') + \frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_1(u)}\partial_{x_1}\delta(u-z')$$

$$+\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_2(u)}\partial_{x_2}\delta(u-z') + ... + \frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_n(u)}\partial_{x_n}\delta(u-z')\Big\} \tag{5.4.25}$$

We give the action on the nth term:

$$\frac{\delta}{\delta Y^{\mu}(z)}\int du\ [\frac{\partial \mathcal{L}[Y(u),Y_n(u)]}{\partial Y^{\mu}_n(u)}\partial_{x_n}\delta(u-z')] = \sum_m \int du\ \frac{\partial^2 \mathcal{L}[Y(u),Y_n(u)]}{\partial Y_{m\mu}(u)\partial Y^{\mu}_n(u)}[\partial_{x_n}\delta(u-z')][\partial_{x_m}\delta(u-z)]$$

Let us include the z,z' integrals:

$$\int dz\ \int dz'\ \dot{G}(z,z')\sum_n\sum_m\int du\ \frac{\partial^2 \mathcal{L}[Y(u),Y_n(u)]}{\partial Y_{m\mu}(u)\partial Y^{\mu}_n(u)}[\partial_{x_n}\delta(u-z')][\partial_{x_m}\delta(u-z)]$$

Now we show that it can be written in the form given in Section 3 so that the rest of the derivation goes through as before. We let the derivative on the delta function act on z,z' instead of u and integrate by parts to get

$$\int dz\ \int dz'\ [\partial_{x_{Bn}}\partial_{x_{Am}}\dot{G}(z,z')]\sum_n\sum_m\int du\ \frac{\partial^2 \mathcal{L}[Y(u),Y_n(u)]}{\partial Y_{m\mu}(u)\partial Y^{\mu}_n(u)}[\delta(u-z')][\partial_{x_m}\delta(u-z)]$$

$$= \int dz\ \int dz'\ [\partial_{x_{Bn}}\partial_{x_{Am}}\dot{G}(z,z')]\sum_n\sum_m\frac{\partial^2 \mathcal{L}[Y(z),Y_n(z)]}{\partial Y_{m\mu}(z)\partial Y^{\mu}_n(z)}\delta(z-z')$$

$$= \int dz\ \ [\langle Y_n(z)Y_m(z)\rangle]\sum_n\sum_m\frac{\partial^2 \mathcal{L}[Y(z),Y_n(z)]}{\partial Y_{m\mu}(z)\partial Y^{\mu}_n(z)}$$

$$= -\sum_{n,m}k_n.k_m\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_n\partial x_m} - \frac{\partial\Sigma}{x_{n+m}}) \tag{5.4.26}$$

Thus we have derived the free equations using the ERG.

5.4.2 Interacting Equations

Two factors of the the functional derivative (5.4.24) given below is what the interaction term is made of.

$$\frac{\delta}{\delta Y^\mu(z')}\int du\ \mathcal{L}(u) = \int du\ \Big\{\frac{\partial\mathcal{L}[Y(u),Y_n(u)]}{\partial Y^\mu(u)}\delta(u-z') - [\partial_{x_1}\frac{\partial\mathcal{L}[Y(u),Y_n(u)]}{\partial Y_1^\mu(u)}]\delta(u-z')$$

$$-[\partial_{x_2}\frac{\partial\mathcal{L}[Y(u),Y_n(u)]}{\partial Y_2^\mu(u)}]\delta(u-z') + ... - [\partial_{x_n}\frac{\partial\mathcal{L}[Y(u),Y_n(u)]}{\partial Y_n^\mu(u)}]\delta(u-z')\Big\} \tag{5.4.27}$$

It can easily be checked that it is *not* gauge invariant.

The resolution of this problem, described in [28, 29], is to introduced all derivatives in the loop variable. The basic idea is that the gauge variation of vertex operators of a given level should be of the form $\lambda_n\frac{\partial}{\partial x_n}$ of lower order vertex operators. This ensures gauge invariance.

We give the results for Spin 1, 2 and then the general result.

Spin 1

$$ik_{1\mu}\frac{\partial Y^\mu}{\partial x_1} \rightarrow \lambda_1\frac{\partial}{\partial x_1}(ik_{0\mu}Y^\mu)$$

$$\frac{\delta}{\delta Y^\mu(z)}\int du\ \mathcal{L}(u) = \frac{\partial\mathcal{L}}{\partial Y^\mu(z)} - \partial_{x_1}\frac{\partial\mathcal{L}}{\partial Y_1^\mu(z)}$$

$$= (ik_{0\mu}\mathcal{L} - ik_{1\mu}ik_{0\nu}Y_1^\nu\mathcal{L})|_{level\ 1} = -[k_{0\mu}k_{1\nu} - k_{1\mu}k_{0\nu}]Y_1^\nu e^{ik_0Y}$$

This is the gauge invariant Maxwell field strength.

Spin 2

$$\mathcal{L} = [iK_{11\mu}\frac{\partial^2Y^\mu}{\partial x_1^2} + iK_{2\mu}\frac{\partial Y^\mu}{\partial x_2} - \frac{1}{2}k_{1\mu}k_{1\nu}Y_1^\mu Y_1^\nu]e^{ik_0Y} \tag{5.4.28}$$

We have introduced $K_{11\mu}, K_{2\mu}$ in place of $k_{2\mu}$ of the free theory. We require

$$\delta K_{2\mu} = \lambda_2 k_{0\mu}\ \ ;\ \delta K_{11\mu} = \lambda_1 k_{1\mu} \tag{5.4.29}$$

This ensures that $\delta(iK_{2\mu}\frac{\partial Y^\mu}{\partial x_2}) = \lambda_2\frac{\partial}{\partial x_2}(ik_0.Y)$ and $\delta(iK_{11\mu}\frac{\partial^2Y^\mu}{\partial x_1^2}) = \lambda_1\frac{\partial}{\partial x_1}(ik_1.Y_1)$

$$K_{2\mu} \equiv (\bar{q}_2 - \frac{\bar{q}_1^2}{2})k_{0\mu}\ \ ;\ \ K_{11\mu} \equiv k_{2\mu} - K_{2\mu} \tag{5.4.30}$$

where $\bar{q}_n \equiv \frac{q_n}{q_0}$, satisfies the required transformation property. It is important to note that if we use $\frac{\partial^2Y^\mu}{\partial x_1^2} = \frac{\partial Y^\mu}{\partial x_2}$, the vertex operators add up to $k_{2\mu}Y_2^\mu$. Also note that dimensional reduction with mass is required for this construction. It has q_0 in the denominator. q_0 is the mass, which therefore has to be non zero. (5.4.30) however is correct even for $\mu = \theta$ when $k_{2\mu} = q_2$ etc.

The quadratic term in the ERG is a product of

$$\frac{\delta}{\delta Y^\mu(z)}\int du\ \mathcal{L}(u) = \frac{\partial\mathcal{L}}{\partial Y^\mu(z)} - \partial_{x_1}\frac{\partial\mathcal{L}}{\partial Y_1^\mu(z)} + \partial_{x_1}^2\frac{\partial\mathcal{L}}{\partial Y_{11}^\mu(z)} - \partial_{x_2}\frac{\partial\mathcal{L}}{\partial Y_2^\mu(z)}$$

at two points z_A and z_B. Let us evaluate this for the modified Lagrangian:

$$\mathcal{L} = [iK_{11\mu}\frac{\partial^2Y^\mu}{\partial x_1^2} + iK_{2\mu}\frac{\partial Y^\mu}{\partial x_2} - \frac{1}{2}k_{1\mu}k_{1\nu}Y_1^\mu Y_1^\nu]e^{ik_0Y} \tag{5.4.31}$$

$$\frac{\partial\mathcal{L}}{\partial Y^\mu} = [ik_{0\mu}iK_{11\nu}\frac{\partial^2Y^\nu}{\partial x_1^2} + ik_{0\mu}iK_{2\nu}\frac{\partial Y^\nu}{\partial x_2} - ik_{0\mu}\frac{1}{2}k_{1\rho}k_{1\nu}Y_1^\nu Y_1^\rho]e^{ik_0Y}$$

$$\partial_{x_1}\frac{\partial\mathcal{L}}{\partial Y_1^\mu} = -k_{1\mu}k_1.Y_2e^{ik_0Y} - k_{1\mu}k_1.Y_1ik_0.Y_1e^{ik_0Y}$$

$$\partial_{x_2}\frac{\partial\mathcal{L}}{\partial Y_2^\mu} = iK_{2\mu}ik_0.Y_2e^{ik_0Y}$$

$$\partial_{x_1}^2\frac{\partial\mathcal{L}}{\partial(\partial_{x_1}^2Y^\mu)} = iK_{11\mu}(ik_0.Y_2+(ik_0.Y_1)^2)e^{ik_0Y}$$

Thus

$$\frac{\partial\mathcal{L}}{\partial X(z)} - \partial_z\frac{\partial\mathcal{L}}{\partial X'(z)} + \partial_z^2\frac{\partial\mathcal{L}}{\partial X''(z)} = \Big(ik_{0\mu}iK_{11\nu}\frac{\partial^2Y^\nu}{\partial x_1^2} + ik_{0\mu}iK_{2\nu}\frac{\partial Y^\nu}{\partial x_2} - ik_{0\mu}\frac{1}{2}k_{1\nu}k_{1\rho}Y_1^\nu Y_1^\rho\Big)e^{ik_0Y}$$
$$+\Big(k_{1\mu}k_1.Y_2e^{ik_0Y} + k_{1\mu}k_1.Y_1ik_0.Y_1e^{ik_0Y}\Big) - iK_{2\mu}ik_0.Y_2e^{ik_0Y}$$
$$+iK_{11\mu}(ik_0.Y_2+(ik_0.Y_1)^2)e^{ik_0Y} \tag{5.4.32}$$

We now replace $\frac{\partial^2Y^\nu}{\partial x_1^2}$ by $\frac{\partial Y^\nu}{\partial x_2}$ and simplify:

The coefficient of Y_2^ν is:

$$V_{2\mu\nu} \equiv \Big(-k_{0\mu}K_{11\nu} - k_{0\mu}K_{2\nu} + k_{1\mu}k_{1\nu} + K_{2\mu}k_{0\nu} - K_{11\mu}k_{0\nu}\Big)e^{ik_0Y} = \Big(-k_{0\mu}K_{11\nu} + k_{1\mu}k_{1\nu} - K_{11\mu}k_{0\nu}\Big)e^{ik_0Y} \tag{5.4.33}$$

The coefficient of $Y_1^\nu Y_1^\rho$ is

$$V_{11\mu\nu\rho} \equiv \Big(-ik_{0\mu}\frac{1}{2}k_{1\nu}k_{1\rho} + i\frac{1}{2}k_{1\mu}(k_{1\nu}k_{0\rho}+k_{1\rho}k_{0\nu}) - iK_{11\mu}k_{0\nu}k_{0\rho}\Big)e^{ik_0Y} \tag{5.4.34}$$

Using ((3.2.23)(5.4.29)) we see that they are invariant.

The components in the θ directions can be obtained from the above. For instance $V_{2\mu\theta}$ is

$$V_{2\mu\theta} = \Big(-k_{0\mu}K_{11\theta} - k_{0\mu}K_{2\theta} + k_{1\mu}q_{1\theta} + K_{2\mu}q_{0\theta} - K_{11\mu}q_{0\theta}\Big)e^{ik_0Y} \tag{5.4.35}$$

5.4.3 General Construction of K's

Let us first introduce the following notation to generalize the construction of the spin 2 and spin 3 cases. Define

$$K_{m\mu}: \delta K_{m\mu} = \lambda_m k_{0\mu}; \quad K_{mn\mu}: \delta K_{mn\mu} = \lambda_m K_{n\mu} + \lambda_n K_{m\mu},\ m\neq n$$

$$K_{mnp\mu}: \delta K_{mnp\mu} = \lambda_m K_{np\mu} + \lambda_n K_{mp\mu} + \lambda_p K_{mn\mu}, \quad m\neq n\neq p \tag{5.4.36}$$

and so on. For repeated indices

$$K_{mm\mu}: \delta K_{mm\mu} = \lambda_m K_{m\mu}; \quad K_{mmm\mu}: \delta K_{mmm\mu} = \lambda_m K_{mm\mu}$$

Also

$$K_{mmp\mu}: \delta K_{mmp\mu} = \lambda_m K_{mp\mu} + \lambda_p K_{mm\mu}$$

and so on.

The general rule is that if $[n]_i$ defines a particular partition of the level N, at which we are working, then

$$\delta K_{[n]_i\mu} = \sum_{m\in[n]_i}\lambda_m K_{[n]_i/m\mu} \tag{5.4.37}$$

where $[n]_i/m$ denotes the partition with m removed, and the sum is over *distinct* m*'s*. (So even if m occurs more than once in the partition, the coefficient of $\lambda_m K_{[n]_i/m\mu}$ is still 1.)

Define

$$\bar{q}(t) \equiv \frac{1}{q_0}q(t) = 1 + \frac{\bar{q}_1}{t} + \frac{\bar{q}_2}{t^2} + ... + \frac{\bar{q}_n}{t^n} + ... \tag{5.4.38}$$

$$= e^{\sum_n y^n t^{-n}} = 1 + \frac{y_1}{t} + \frac{y_2 + \frac{y_1^2}{2}}{t^2} + \frac{y_3 + y_1 y_2 + \frac{y_1^3}{6}}{t^3} + ...$$

If we solve for y_n in terms of q_m we get

$$\bar{q}_1 = y_1; \quad \bar{q}_2 = y_2 + \frac{y_1^2}{2} \implies y_2 = \bar{q}_2 - \frac{\bar{q}_1^2}{2};$$

Similarly

$$y_3 = \bar{q}_3 - \bar{q}_2\bar{q}_1 + \frac{\bar{q}_1^3}{3}$$

In general $\sum_{n=0}^{\infty} \frac{y_n}{t^n} = ln\ (\bar{q}(t))$.

Similarly define

$$\lambda(t) = 1 + \frac{\lambda_1}{t} + ...\frac{\lambda_n}{t^n} + ... = e^{\sum_0^\infty z_n t^{-n}}$$

The gauge transformation $\bar{q}(t) \to \lambda(t)\bar{q}(t)$ is represented as $y_n \to y_n + z_n$. Since we are only interested in the lowest order in λ we can set $z_n = \lambda_n$. Thus we have

$$\delta y_n = \lambda_n \tag{5.4.39}$$

.

Let us now construct the $K_{mnp..\mu}$: Let us start by defining $K_{0\mu} \equiv k_{0\mu}$. Then $K_{1\mu} = k_{1\mu}$, because $\delta K_{0\mu} = \lambda_1 K_{0\mu}$.

Level 2:

Let

$$K_{2\mu} = y_2 k_{0\mu} \tag{5.4.40}$$

Using (5.4.39), it clearly satisfies the requirement (5.4.36), that $\delta K_{2\mu} = \lambda_2 K_{0\mu}$. Using $y_2 = \bar{q}_2 - \frac{\bar{q}_1^2}{2}$,

$$K_{2\mu} = (\bar{q}_2 - \frac{\bar{q}_1^2}{2})k_{0\mu}$$

Then we let

$$K_{11\mu} = k_{2\mu} - K_{2\mu} \tag{5.4.41}$$

It is easy to check that $\delta K_{11\mu} = \lambda_1 K_{1\mu}$.

We can now generalize this construction:

$\boldsymbol{K_n,\ n \geq 2}$:

Consider $K_{n\mu}$. Since we want $\delta K_{n\mu} = \lambda_n K_{0\mu}$, a correct choice is

$$K_{n\mu} = y_n k_{0\mu} \tag{5.4.42}$$

$\boldsymbol{K_{n1\mu},\ n \geq 2:}$

We need $\delta K_{n1\mu} = \lambda_1 K_{n\mu} + \lambda_n K_{1\mu}$. An obvious trial is to set

$$K_{n1\mu} = y_n K_{1\mu} = y_n k_{1\mu} \tag{5.4.43}$$

Using (5.4.55,5.4.39) we see that it is correct.

$\boldsymbol{K_{mn\mu},\ m \neq n; n, m \geq 2:}$

It is easy to check that

$$K_{mn\mu} = y_n y_m k_{0\mu} \tag{5.4.44}$$

satisfies $\delta K_{nm\mu} = \lambda_n y_m k_{0\mu} + \lambda_m y_n k_{0\mu} = \lambda_n K_{m\mu} + \lambda_m K_{n\mu}$ as required.

$\boldsymbol{K_{mm..\mu},\ m \geq 2:}$

For repeated indices we see that what works is:

$$K_{mm\mu} = \frac{y_m^2}{2} k_{0\mu}; \quad K^{\mu}_{mmm} = \frac{y_m^3}{3!} k_{0\mu}; \tag{5.4.45}$$

The pattern is easily generalized.

$\boldsymbol{K_{mn1\mu},\ m \neq n;\ m, n \geq 2:}$

$$K_{mn1\mu} = y_n y_m K_{1\mu} \tag{5.4.46}$$

Satisfies

$$\delta K_{mn1\mu} = \lambda_n y_m K_{1\mu} + \lambda_m y_n K_{1\mu} + \lambda_1 y_n y_m k_{0\mu} = \lambda_n K_{m1\mu} + \lambda_m K_{n1\mu} + \lambda_1 K_{mn\mu}$$

as required.

Again for repeated indices:

$$K_{mm1\mu} = \frac{y_m^2}{2} K_{1\mu} \tag{5.4.47}$$

$\boldsymbol{K_{n11\mu},\ n \geq 2:}$

We try

$$K_{n11\mu} = y_n K_{11\mu} \tag{5.4.48}$$

$\delta K_{n11\mu} = \lambda_n K_{11\mu} + \lambda_1 y_n K_{1\mu} = \lambda_n K_{11\mu} + \lambda_1 K_{n1\mu}$ as required.

The pattern is now clear: when all the $m, n, .. \geq 2$ we just get

$\boldsymbol{K_{mn...\mu},\ m \neq n;\ n \geq 2:}$

$$K_{mn..\mu} = y_m y_n ... k_{0\mu} \tag{5.4.49}$$

$\boldsymbol{K_{mn...1\mu},\ n \geq 2:}$

When one of the indices is 1, we get

$$K_{mn..1\mu} = y_m y_n ... k_{1\mu} \tag{5.4.50}$$

$\boldsymbol{K_{mn...11\mu},\ n \geq 2:}$ Similarly if two of the indices are 1 we get

$$K_{mn..11\mu} = y_m y_n ... K_{11\mu} \tag{5.4.51}$$

$\boldsymbol{K_{m.....11\mu},\ n \geq 2:}$

$$K_{m\mu\underbrace{1111..1}_{n}} = y_m K_{\underbrace{1111..1\mu}_{n}} \tag{5.4.52}$$

For other repeated indices again the pattern is obvious. Thus

$$K_{mm\underbrace{111...\mu}_{n}} = \frac{y_m^2}{2} K_{\underbrace{111..\mu}_{n}} \tag{5.4.53}$$

$\boldsymbol{K_{\underbrace{1.....11\mu}_{n}}:}$

To complete the construction we need $K_{111..1\mu}$. For the second level we had $K_{11\mu} = k_{2\mu} - K_{2\mu}$. Similarly one can check that

$$K_{111\mu} = k_{3\mu} - K_{21\mu} - K_{3\mu}$$

$\delta K_{111\mu} = \lambda_3 k_{0\mu} + \lambda_2 k_{1\mu} + \lambda_1 k_{2\mu} - \lambda_2 K_{1\mu} - \lambda_1 K_{2\mu} - \lambda_3 k_{0\mu} = \lambda_1 (k_{2\mu} - K_{2\mu}) = \lambda_1 K_{11\mu}$ as required.

It is natural to try

$$K_{\underbrace{1....1\mu}_{n}} = k_{n\mu} - \sum_{[n]_i \in [n]'} K_{[n]_i \mu} \tag{5.4.54}$$

where $[n]'$ indicates all the partitions of n *except* $\underbrace{1...1}_{n}$.

We prove this by recursion:

$$K_{[n]\mu} \equiv \sum_{[n]_i \in [n]} K_{[n]_i\mu} = k_{n\mu} \tag{5.4.55}$$

Proof:

Let us assume that the above is true for n. Consider $K_{[n+1]'_i\mu}$. We have

$$\delta K_{[n+1]'_i\mu} = \sum_{m \in [n+1]'_i} \lambda_m K_{[n+1]'_i/m\mu}$$

The sum, as always, is over distinct m's. This is true because such K's have all been explicitly constructed for all n.

For e.g. let us explicitly write out the coefficient of λ_2 in the above equation - it is $\lambda_2 K_{[n+1]'_i/2}$. Thus we can write

$$\delta K_{[n+1]'_i\mu} = \lambda_1 K_{[n+1]'_i/1\mu} + \lambda_2 K_{[n+1]'_i/2\mu} + \lambda_3 K_{[n+1]'_i/3\mu} + ...$$

Note that $[n+1]'_i/2$ is a partition of $n+1$ with one 2 removed. If we sum over all i this gives all the partitions of $n+1$ with one 2 removed, i.e. *all partitions of* $n-1$ i.e. $[n-1]$. Similarly $[n+1]'_i/3$ summed over all i gives all partitions of $n-2$, i.e. $[n-2]$. However $[n+1]'_i/1$ gives all partitions of n *except for the one with all one's*,i.e. it gives $[n]'$. Now sum over i and note that the LHS is $\sum_i K_{[n+1]'_i}$ and has all the K's at this level except for $K_{\underbrace{1...1}_{n+1}}$.

Thus

$$\sum_i \delta K_{[n+1]'_i\mu} = \lambda_1 K_{[n]'\mu} + \lambda_2 K_{[n-1]\mu} + \lambda_3 K_{[n-2]\mu} + ..\lambda_m K_{[n+1-m]\mu}.$$

Using (5.4.55) we see that this becomes

$$\sum_i \delta K_{[n+1]'_i\mu} = \lambda_1 K_{[n]'\mu} + \lambda_2 k_{(n-1)\mu} + \lambda_3 k_{(n-2)\mu} + ..\lambda_m k_{(n+1-m)\mu} + ...$$

$$= (\lambda_1 (K_{[n]^\mu} - K_{\underbrace{1...1}_{n}\mu}) + \lambda_2 k_{n-1\mu} + \lambda_3 k_{n-2\mu} + ..\lambda_m k_{n+1-m\mu} + ...$$

$$= (\lambda_1 k_{n\mu} - \delta K_{\underbrace{1...1\mu}_{n+1}} + \lambda_2 k_{(n-1)\mu} + \lambda_3 k_{(n-2)\mu} + ..\lambda_m k_{(n+1-m)\mu} + ...$$

So

$$\sum_i \delta K_{[n+1]_i\mu} = \delta k_{(n+1)\mu}$$

Thus

$$K_{[n+1]\mu} = k_{n\mu}$$

Since it is true for $n = 2, 3$ this completes the proof.

This completes the construction of K's for all levels. There is one more step to be taken before we can use these to write down an interaction term in string theory. This is to perform a truncation of the spectrum using the Q-rules which have been worked out up to level 5. This can be implemented on all the terms in the EOM. This is a fairly mechanical step and we do not do it in this review.

Level 3

We can now construct the level 3 gauge invariant field strength using these formulae:

$$K_{3\mu} = y_3 k_{0\mu} = (\bar{q}_3 - \bar{q}_2\bar{q}_1 + \frac{\bar{q}_1^3}{3})k_{0\mu}$$

$$K_{21\mu} = y_2 k_{1\mu} = (\bar{q}_2 - \frac{\bar{q}_1^2}{2})k_{1\mu}$$

$$K_{111\mu} = k_{3\mu} - K_{21\mu} - K_{3\mu}$$

The level three part of the exponent of $\mathcal{L}$ is

$$K_{3\mu}\frac{\partial Y^\mu}{\partial x_3} + K_{21\mu}\frac{\partial^2 Y^\mu}{\partial x_2 \partial x_1} + K_{111\mu}\frac{\partial^3 Y^\mu}{\partial x_1^3}$$

and the full Lagrangian at Level 3 is ($Y_n^\mu \equiv \frac{\partial Y^\mu}{\partial x_n}$):

$$\mathcal{L} = [iK_{3\mu}Y_3^\mu + iK_{21\mu}\frac{\partial^2 Y^\mu}{\partial x_2 \partial x_1} + iK_{111\mu}\frac{\partial^3 Y^\mu}{\partial x_1^3} - K_{2\mu}K_{1\nu}Y_2^\mu Y_1^\nu$$

$$-K_{11\mu}K_{1\nu}\frac{\partial^2 Y^\mu}{\partial x_1^2}Y_1^\nu - i\frac{k_{1\mu}k_{1\nu}k_{1\rho}}{3!}Y_1^\mu Y_1^\nu Y_1^\rho]e^{ik_0 Y} \tag{5.4.56}$$

We evaluate the functional derivative

$$\frac{\delta}{\delta Y^\mu(z')}\int du\ \mathcal{L}(u)$$

to obtain:

$$L_{3\mu(z)} \equiv \Big[V_{3\mu\nu}Y_3^\nu(z) + V_{21\mu\rho\sigma}Y_2^\rho(z)Y_1^\sigma(z) + V_{111\mu\lambda\rho\sigma}Y_1^\lambda(z)Y_1^\rho(z)Y_1^\sigma(z)\Big]e^{ik_0.Y(z)}$$

where

$$\begin{aligned}
V_{3\mu\rho} &= -k_{0\mu}[K_{3\rho} + K_{21\rho} + K_{111\rho}] + k_{1\mu}[K_{11\rho} + K_{2\rho}] + K_{2\mu}k_{1\rho} - K_{11\mu}k_{1\rho} - K_{21\mu}k_{0\rho} + K_{111\mu}k_{0\rho} + K_{3\mu}k_{0\rho} \\
V_{21\mu\rho\sigma} &= i\Big[- k_{0\mu}K_{11\rho}k_{1\rho} + k_{1\mu}K_{11\rho}k_{0\sigma} + k_{1\mu}K_{2\rho}k_{0\sigma} + k_{1\mu}k_{1\rho}k_{1\sigma} \\
&\quad -2K_{11\mu}k_{1\rho}k_{0\sigma} - K_{11\mu}k_{0\rho}k_{1\sigma} - K_{21\mu}k_{0\rho}k_{0\sigma} + 3K_{111\mu}k_{0\rho}k_{0\sigma}\Big] \\
V_{111}^{\mu\lambda\rho\sigma} &= \frac{1}{3!}k_{0\mu}k_{1\lambda}k_{1\rho}k_{1\sigma} - \frac{1}{3!}k_{1\mu}k_{1(\lambda}k_{1\rho}k_{0\sigma)} + \frac{1}{3}K_{11\mu}k_{1(\lambda}k_{0\rho}k_{0\sigma)} - K_{111\mu}k_{0\lambda}k_{0\rho}k_{0\sigma}
\end{aligned} \tag{5.4.57}$$

$L_{3\mu}(z)$ is a gauge invariant field strength for the massive level 3 fields. Note once again that the non-zero mass (q_0) is crucial for being able to construct such an object.

5.4.4 Interacting Equations of Motion

The interacting part of ERG can be written as a sum of terms of the form given below. It is of the form: Field strength $\times$ Field strength. The fact that the equations of motion are just quadratic is not surprising given that the basic vertex involves splitting and joining of strings.

However an interesting point is that the field strengths are gauge invariant under the *same* transformations as that of the free theory. This is unlike non Abelian gauge theories, where the transformation law is modified when the coupling constant is non zero. In this sense the equations are Abelian. Note also that if Chan Paton factors are included to make the low energy sector Yang Mills instead of electromagnetism, then the gauge transformation laws would be modified even in the loop variable formalism.

For example an interaction involving two V_3's given above is:

$$\int dz \int dz'\dot{G}(z,z')\eta^{\mu\nu}V_{3\mu\rho}Y_3^\rho(z)e^{ik_{0A}.Y(z)}V_{3\nu\sigma}Y_3^\sigma(z')e^{ik_{0B}.Y(z')}$$

In all these equations we have not made any restriction on the range of μ. The construction superficially works for $\mu = \theta$ also. However we need some restriction on the Green function $G^{\theta\theta}(z,z')$ if we are to reproduce string theory S-matrix. We certainly do not want the structure of the Veneziano amplitude to be modified. Thus the simplest solution is to set $G^{\theta\theta}(z,z') = 0$ when $z \neq z'$ and leave it unchanged for $z = z'$

because the free equations require that. As a constructive algorith this is fine. Whether we can deduce this from first principles is an open problem.

In this BRST string field theory superficially looks non Abelian even when the gauge group for the massless gauge field is $U(1)$. The gauge transformation law of the massless photon has extra pieces due to interactions. However it is possible that there may be field redefinitions that get rid of these. There is some hint about this possibility in [43].

6 Curved Space Time

6.1 Map to curved space time: problem of gauge invariance

Thus far we have been working in flat space time. The loop variable expressions are mapped in a straightforward way to space time fields and their derivatives. Expressions that are gauge invariant when written in terms of loop variables, continue to be gauge invariant after the map to space time fields. If our aim is a manifestly background independent formalism then we must learn how to apply all this in curved space time. This issue will become more acute when we deal with closed string theory in the next section. So it is appropriate to sort this issue out first. Using Riemann Normal Coordinates (RNC) is the first step towards solving this problem.

The loop variable method gives us equations in momentum space. Thus writing

$$f(x) = \int dk\; e^{ikx} f(k) = \int dk\; (1 + ik.x + \frac{(ik.x)^2}{2!} + ...) f(k) = f(0) + x\partial f(0) + \frac{x^2}{2!}\partial^2 f(0) + ... \tag{6.1.1}$$

we see that powers of k correspond to terms in the Taylor expansion. Thus we need to do a Taylor expansion in curved space time. This is best done using Riemann Normal Coordinates [44, 45, 46]. Appendix A (A) contains a summary of all the aspects of the Riemann Normal Coordinate system that are needed for this paper. Note that dimensional reduction has to be done before we talk about curved space time. So μ ranges from 0 to D-1.

Consider a loop variable expression $k_{0\mu}k_{1\rho}k_{1\sigma}$. In curved space time we work in the Riemann Normal Coordinate (RNC) system and interpret $ik_{0\mu} \approx \frac{\partial}{\partial \bar{Y}^\mu}$ where $\bar{Y}^\mu$ are RNC's. Then the map to space time fields involves integrating over $\Psi[k_0, k_1, ..\lambda_1, ..]$ and is:

$$\langle k_{1\rho}k_{1\sigma}\rangle = S_{11\rho\sigma}(k_0) \quad ; \int dk_0 e^{ik_0.\bar{Y}(z)} S_{11\rho\sigma}(k_0) = S_{11\rho\sigma}(\bar{Y}(z))$$

Thus

$$\int dk_0 [1 + ik_{0\mu}\bar{Y}^\mu(z) + ...] S_{11\rho\sigma}(k_0) = S_{11\rho\sigma}(0) + \partial_\mu S_{11\rho\sigma}(0)\bar{Y}^\mu(z) + ... \tag{6.1.2}$$

Thus we can conclude that the loop variable expression $k_{0\mu}k_{1\rho}k_{1\sigma}$ gets mapped to $\partial_\mu S_{11\rho\sigma}(0)$ which is the first term in the Taylor expansion of the vector. Similarly $k_{0\mu}k_{0\nu}k_{1\rho}k_{1\sigma}$ gets mapped to $\partial_\mu\partial_\nu S_{11\rho\sigma}(0)$, the second term in the Taylor expansion etc. We also know from (A.3.17) of the Appendix A (A), how to write these Taylor expansion coefficients in a manifestly covariant form.

Thus for instance

$$\langle k_{0\mu}k_{1\rho}k_{1\sigma}\rangle = \nabla_\mu S_{11\rho\sigma}(0)$$

and

$$\langle k_{0\mu}k_{0\nu}k_{1\rho}k_{1\sigma}\rangle = \{\nabla_\nu\nabla_\mu S_{\rho\sigma}(0) \;-\; \frac{1}{3}(R^\beta{}_{\mu\rho\nu}(0)S_{\beta\sigma}(0) + R^\beta{}_{\mu\sigma\nu}(0)S_{\beta\rho}(0))\}$$

Note that the LHS is manifestly symmetric in μ,ν. Since covariant derivatives do not commute, the first term is not symmetric. The role of the second term is to compensate for this and make the RHS also symmetric in μ,ν. Thus this method gives us a map from Loop Variable expressions to covariant expressions in curved space time. However as we will see below, this map does not commute with gauge transformation. Let $\mathcal{L}$ be

the set of loop variable expressions that gets mapped to the set $\mathcal{S}$, of expressions involving space time fields. Let us call this map $\mathcal{M}$. Thus

$$\mathcal{M} : k_{0\mu}k_{1\rho}k_{1\sigma} \in \mathcal{L} \rightarrow \langle k_{0\mu}k_{1\rho}k_{1\sigma}\rangle = \nabla_\mu S_{11\rho\sigma} \in \mathcal{S} \tag{6.1.3}$$

Now one can perform a gauge transformation g on the loop variable expression and then map it to space time fields by $\mathcal{M}$. This can be compared with the gauge transformation of the expression involving space time fields. The question is whether the result is the same whether we go from $\mathcal{L}$ to $\mathcal{S}^g$ along either path.

$$\begin{array}{lll} \mathcal{M}: & \mathcal{L} & \longrightarrow \mathcal{S} \\ & g\downarrow & \quad g\downarrow \\ \mathcal{M}: & \mathcal{L}^g & \longrightarrow \mathcal{S}^g \end{array} \tag{6.1.4}$$

We know the gauge transformation law of $S_{11\rho\sigma}$:

$$g : S_{11\rho\sigma} \rightarrow S_{11\rho\sigma} + \nabla_{(\rho}\Lambda_{11\sigma)}$$

Therefore

$$g : \nabla_\mu S_{11\rho\sigma} \rightarrow \nabla_\mu S_{11\rho\sigma} + \nabla_\mu\nabla_{(\rho}\Lambda_{11\sigma)} \tag{6.1.5}$$

Simlarly

$$g : k_{0\mu}k_{1\rho}k_{1\sigma} \rightarrow k_{0\mu}k_{1\rho}k_{1\sigma} + k_{0\mu}(\lambda_1 k_{0\rho}k_{1\sigma} + \lambda_1 k_{1\sigma}k_{0\rho})$$

We now have to compare (6.1.5)

$$\mathcal{M} : k_{0\mu}(\lambda_1 k_{0\rho}k_{1\sigma} + \lambda_1 k_{1\sigma}k_{0\rho}) \rightarrow \{\nabla_\mu\nabla_{(\rho}\Lambda_{11\sigma)}(0) \ - \ \frac{1}{3}R^{\beta}{}_{(\rho\sigma)\mu}(0)\Lambda_{11\beta}(0)\} \tag{6.1.6}$$

We now have to compare (6.1.5) with (6.1.6) obtained by the other path. We see that they differ by terms proportional to the curvature tensor. Thus while in flat space they agree, in curved space they don't. The non commutativity of this process has the consequence that loop variable expressions that are gauge invariant do not get mapped to gauge invariant expressions when mapped to space time fields. In a nutshell this is because $k_{0\mu}k_{0\nu} = k_{0\nu}k_{0\mu}$ but $\nabla_\mu\nabla_\nu \neq \nabla_\nu\nabla_\mu$.

6.2 Prescription

6.2.1 Motivating the prescription

We now give a well defined constructive algorithm for mapping to space time in such a way that gauge invariance is maintained. This was described in [35]. The prescription involves q_n in a crucial way and so dimensional reduction with mass is important. In this section μ will range from 0 to D-1.

We have seen the problem arises when gauge transformation produces an extra derivative in the form $k_n \rightarrow k_n + \lambda_n k_0 + ...$ The problem is solved by rewriting the loop variable expression in such a way that no extra derivative terms appear in any gauge transformation. All derivatives lurking in gauge transformations are made manifest right away. This can be implemented as follows: Define

$$k_{n\mu} = \tilde{k}_{n\mu} + y_n k_{0\mu} \tag{6.2.7}$$

where $y_n \rightarrow y_n + \lambda_n$ under a gauge transformation. y_n have been defined earlier [29]:

$$\sum_{n=0} q_n t^{-n} = q_0 e^{\sum_{m=1} y_m t^{-m}} \tag{6.2.8}$$

Gauge transformation of $k_{n\mu}$ is given by

$$k_{n\mu} \rightarrow k_{n\mu} + \lambda_1 k_{n-1\mu} + \lambda_2 k_{n-2\mu} + ...\lambda_{n-1}k_{1\mu} + \lambda_n k_{0\mu} \tag{6.2.9}$$

Thus $\tilde{k}_{n\mu}$ satisfies a gauge transformation rule

$$\tilde{k}_{n\mu} \to \tilde{k}_{n\mu} + \lambda_1 k_{n-1\mu} + \lambda_2 k_{n-2\mu} + ...\lambda_{n-1} k_{1\mu} \tag{6.2.10}$$

Once the loop variable expressions are written in terms of tilde variables, gauge transformations do not produce any new derivatives. Thus all the required curvature couplings are present right in the beginning. Now we can map these to new space time fields (also with tildes). This ensures that the map to space time fields commutes with gauge transformations. Expressions that were gauge invariant at the level of loop variables, continues to be gauge invariant in terms of the tilde space time fields. The tilde fields are expressible in terms of the original fields - these are just field redefinitions. Once we have a covariant gauge invariant expression in terms of tilde space time fields, we can undo the field redefinitions. Field redefinitions done on the space time fields do not change any of the gauge transformation properties.

6.2.2 Illustration of Procedure

Consider the level 2 field

$$\langle k_{1\mu} k_{1\nu} \rangle = S_{11\mu\nu} \tag{6.2.11}$$

We define tilde variables by

$$k_{1\mu} = \tilde{k}_{1\mu} + y_1 k_{0\mu} \tag{6.2.12}$$

Then

$$k_{1\mu} k_{1\nu} = \tilde{k}_{1\mu}\tilde{k}_{1\nu} + k_{0\mu} y_1 \tilde{k}_{1\nu} + k_{0\nu} y_1 \tilde{k}_{1\mu} + y_1^2 k_{0\mu} k_{0\nu} \tag{6.2.13}$$

Define tilde space time fields by

$$\begin{aligned}
\langle \tilde{k}_{1\mu}\tilde{k}_{1\nu} \rangle &= \tilde{S}_{11\mu\nu} \\
\langle y_1 \tilde{k}_{1\nu} \rangle &= \tilde{S}_{11\nu} \\
\langle y_1^2 \rangle &= \tilde{S}_{11}
\end{aligned} \tag{6.2.14}$$

The relations between the old space time fields and the tilde fields are given by:

$$S_{11\mu\nu} = \tilde{S}_{11\mu\nu} + \nabla_{(\mu}\tilde{S}_{11\nu)} + \nabla_\mu \nabla_\nu \tilde{S}_{11} \tag{6.2.15}$$

$$\tilde{S}_{11\mu} = \frac{S_{11\mu}}{q_0} - \frac{1}{q_0^2}\nabla_\mu S_{11} \quad \tilde{S}_{11} = \frac{S_{11}}{q_0^2} \tag{6.2.16}$$

where

$$\langle q_1 k_{1\mu} \rangle = S_{11\mu}; \qquad \langle q_1^2 \rangle = S_{11} \tag{6.2.17}$$

and also $q_1 = y_1 q_0$.

Let us turn to the gauge transformation laws for these fields: Using $k_{1\mu} \to k_{1\mu} + \lambda_1 k_{0\mu}$, $y_1 \to y_1 + \lambda_1$, we obtain

$$\begin{aligned}
\delta \tilde{S}_{11\mu\nu} &= 0 \\
\delta \tilde{S}_{11\mu} &= \langle \lambda_1 \tilde{k}_{1\mu} \rangle = \tilde{\Lambda}_{11\mu} \\
\delta \tilde{S}_{11} &= 2\langle y_1 \lambda_1 \rangle = 2\tilde{\Lambda}_{11}
\end{aligned} \tag{6.2.18}$$

We see that there are no derivatives in the transformation laws.

If we define

$$\langle \lambda_1 k_{1\mu} \rangle = \Lambda_{11\mu} = \langle \lambda_1 \tilde{k}_{1\mu} + \lambda_1 y_1 k_{0\mu} \rangle = \tilde{\Lambda}_{11\mu} + \nabla_\mu \tilde{\Lambda}_{11} \tag{6.2.19}$$

we see that combining (6.2.15), (6.2.16) and (6.2.18) gives:

$$\begin{aligned}
\delta S_{11\mu\nu} &= \nabla_{(\mu}\Lambda_{11\nu)} \\
\delta S_{11\mu} &= \nabla_\mu \Lambda_{11} + q_0 \Lambda_{11\mu} \\
\delta S_{11} &= 2\Lambda_{11} q_0
\end{aligned} \tag{6.2.20}$$

This is of course the correct gauge transformation for the original space time fields. This illustrates the (obvious) fact that the above field redefinitions do not introduce any inconsistencies in gauge transformations. This has to do with the fact that gauge transformations of the tilde fields do not involve derivatives.

Let us now apply this prescription to the loop variable expression considered earlier:

$$k_{0\mu}k_{1\rho}k_{1\sigma} \tag{6.2.21}$$

In terms of tilde variables it is

$$k_{0\mu}(\tilde{k}_{1\rho}\tilde{k}_{1\sigma} + k_{0\rho}y_1\tilde{k}_{1\sigma} + k_{0\sigma}y_1\tilde{k}_{1\rho} + k_{0\rho}k_{0\sigma}y_1^2) \tag{6.2.22}$$

Mapping to space-time fields in the usual way (described in the last subsection) we get

$$\nabla_\mu\tilde{S}_{11\rho\sigma}+\nabla_\mu\nabla_\rho\tilde{S}_{11\sigma}+\nabla_\mu\nabla_\sigma\tilde{S}_{11\rho}+\frac{2}{3}(R^\beta{}_{\rho\mu\sigma}+R^\beta{}_{\sigma\mu\rho})\tilde{S}_{11\beta}+\nabla_\mu\nabla_\rho\nabla_\sigma\tilde{S}_{11}+\frac{1}{3}(R^\beta{}_{\rho\mu\sigma}+R^\beta{}_{\sigma\mu\rho})\nabla_\beta\tilde{S}_{11} \tag{6.2.23}$$

Since the gauge transformation of the loop variable expression (6.2.22) does not produce any extra $k_{0\mu}$ its space-time map is guaranteed to coincide with the gauge transformation of (6.2.23). Thus we have an internally self consistent prescription for mapping to space-time fields.

The last step of the prescription is to rewrite (6.2.23) in terms of our old space time fields. This does not modify the gauge transformation properties of this expression. We use the field redefinitions (6.2.15),(6.2.16). The two and three derivative term cancel. This is clear because these terms do not involve the curvature tensor and one might as well have been in flat space. In flat space the original expression had only one space time derivative. Performing some field redefinitions and then undoing them gets you back to the starting point.

$$\nabla_\mu S_{11\rho\sigma} + \frac{2}{3}(R^\beta{}_{\rho\mu\sigma} + R^\beta{}_{\sigma\mu\rho})[\frac{S_{11\beta}}{q_0} - \frac{1}{2}\nabla_\beta\frac{S_{11}}{q_0^2}] \tag{6.2.24}$$

Now we can compare the gauge transforms of (6.2.21) and (6.2.24) to verify that they agree. The gauge variation of (6.2.21) is

$$k_{0\mu}(k_{0\rho}\lambda_1 k_{1\sigma} + k_{0\sigma}\lambda_1 k_{1\rho}) \tag{6.2.25}$$

Mapping (6.2.25) to space-time fields gives

$$\nabla_\mu(\nabla_\rho\Lambda_{11\sigma} + \nabla_\sigma\Lambda_{11\rho}) + \frac{2}{3}(R^\beta{}_{\rho\mu\sigma} + R^\beta{}_{\sigma\mu\rho})\Lambda_{11\beta} \tag{6.2.26}$$

The gauge variation of (6.2.24) is (using (6.2.20)) seen to be the same as above.

6.2.3 Summary of prescription

We summarize the prescription below:

1. Step 1: Define tilde variables by $k_{n\mu} = \tilde{k}_{n\mu} + y_n k_{0\mu}$. Their gauge transformation law has no $k_{0\mu}$. Rewrite all loop variable expressions in terms of tilde variables.

2. Step 2: Define new space time fields by mapping the tilde variables to new (tilde) space time fields using the map $\mathcal{M}$. Obtain the relation between the old space time fields and the new ones.

3. Step 3: Map all loop variable expressions (now written in terms of tilde variables) to expressions involving the new space time tilde fields by the same map $\mathcal{M}$. At this point a gauge invariant expression involving loop variables is mapped to a gauge invariant expression involving tilde space time fields. This expression may involve higher derivatives than the expression we started with.

4. Step 4: Rewrite the tilde space time fields in terms of the old space time fields. All the higher derivative terms will cancel. We get an expression involving space time fields which has the naive covariantization, plus some extra curvature couplings to Stuckelberg fields and derivatives thereof.

6.3 Example: Open String Level 2

The free equation of motion is [23, 28, 29]:

$$-k_1.k_1 ik_0 Y_2 - \frac{1}{2}k_1.k_1(ik_0.Y_1)^2 - k_1.k_0(k_1.Y_1)(k_0.Y_1)$$

$$+\frac{1}{2}k_0^2(k_1.Y_1)^2 - k_0^2 ik_2.Y_2 + ik_1.k_0(k_1.Y_2) + k_2.k_0 ik_0.Y_2 = 0 \tag{6.3.27}$$

The coefficient of $Y_1^\mu Y_1^\nu$ is

$$k_0^2 k_{1\mu}k_{1\nu} - k_0.k_1 k_{1(\mu}k_{\nu)0} + k_1.k_1 k_{0\mu}k_{0\nu} = 0 \tag{6.3.28}$$

This is gauge invariant under $k_{1\mu} \to k_{1\mu} + \lambda_1 k_{0\mu}$. Dimensional reduction gives a massive field with EOM:

$$(k_0^2 + q_0^2)k_{1\mu}k_{1\nu} - (k_0.k_1 + q_0 q_1)k_{1(\mu}k_{\nu)0} + (k_1.k_1 + q_1 q_1)k_{0\mu}k_{0\nu} = 0 \tag{6.3.29}$$

This has to be mapped to space time fields.

Let us apply this procedure to a general loop variable expression

$$k_{0\mu}k_{0\nu}k_{1\rho}k_{1\sigma} \tag{6.3.30}$$

All the terms in (6.3.29) involving the tensor field can be obtained from this by contractions.

Step 1

We let $k_{1\mu} = \tilde{k}_{1\mu} + y_1 k_{0\mu}$. Then (6.3.30) becomes

$$k_{0\mu}k_{0\nu}k_{1\rho}k_{1\sigma} = k_{0\mu}k_{0\nu}(\tilde{k}_{1\rho}\tilde{k}_{1\sigma} + \tilde{k}_{1\rho}y_1 k_{0\sigma} + k_{0\rho}y_1\tilde{k}_{1\sigma} + y_1^2 k_{0\rho}k_{0\sigma}) \tag{6.3.31}$$

Step 2 has already been done - the required tilde space time fields were defined there in (6.2.14).

Step3

Let us consider each term in turn and map to space time fields, using the definitions (6.2.14) and the map M. We get

$$\begin{aligned}
\langle k_{0\mu}k_{0\nu}\tilde{k}_{1\rho}\tilde{k}_{1\sigma}\rangle &= \frac{1}{2}[\nabla_{(\mu}\nabla_{\nu)}\tilde{S}_{11\rho\sigma} + \frac{1}{3}R^\beta{}_{(\mu\nu)\rho}\tilde{S}_{11\beta\sigma} + \frac{1}{3}R^\beta{}_{(\mu\nu)\sigma}\tilde{S}_{11\rho\beta}] \\
\langle k_{0\mu}k_{0\nu}k_{0\sigma}\tilde{k}_{1\rho}y_1\rangle &= \frac{1}{6}[\nabla_{(\mu}\nabla_\nu\nabla_{\sigma)}\tilde{S}_{11\rho} - R^\beta{}_{(\mu|\rho|\nu}\nabla_{\rho)}\tilde{S}_{11\beta} - \frac{1}{2}\nabla_{(\sigma}R^\beta{}_{\mu|\rho|\nu)}\tilde{S}_{11\beta}] \\
\langle k_{0\mu}k_{0\nu}k_{0\rho}k_{0\sigma}y_1^2\rangle &= \frac{1}{4!}\nabla_{(\mu}\nabla_\nu\nabla_\rho\nabla_{\sigma)}\tilde{S}_{11}
\end{aligned} \tag{6.3.32}$$

The gauge transformations of the tilde variables are given in (6.2.18) and do not involve derivatives. Thus in (6.3.32) the gauge transformations of the LHS and RHS are guaranteed to agree. Now we proceed to Step 4:

Step 4

Now we can redefine fields in terms of the original fields using (6.2.15) and (6.2.16):

$$\begin{aligned}
\langle k_{0\mu}k_{0\nu}\tilde{k}_{1\rho}\tilde{k}_{1\sigma}\rangle &= \frac{1}{2}\Big(\nabla_{(\mu}\nabla_{\nu)}[S_{11\rho\sigma} - \frac{\nabla_{(\rho}S_{11\sigma)}}{q_0^2} + \frac{\nabla_\rho\nabla_\sigma S_{11}}{q_0^2}] + \\
&\quad \frac{1}{3}R^\beta{}_{(\mu\nu)\rho}[S_{11\beta\sigma} - \frac{\nabla_{(\beta}S_{11\sigma)}}{q_0^2} + \frac{\nabla_\beta\nabla_\sigma S_{11}}{q_0^2}] + \frac{1}{3}R^\beta{}_{(\mu\nu)\sigma}[S_{11\rho\beta} - \frac{\nabla_{(\rho}S_{11\beta)}}{q_0^2} + \frac{\nabla_\rho\nabla_\beta S_{11}}{q_0^2}]\Big) \\
\langle k_{0\mu}k_{0\nu}k_{0\sigma}\tilde{k}_{1\rho}y_1\rangle &= \frac{1}{6}\Big(\nabla_{(\mu}\nabla_\nu\nabla_{\rho)}[\frac{S_{11\sigma}}{q_0} - \frac{\nabla_\sigma S_{11}}{q_0^2}] - R^\beta{}_{(\mu|\sigma|\nu}\nabla_{\rho)}[\frac{S_{11\beta}}{q_0} - \frac{\nabla_\beta S_{11}}{q_0^2}] - \frac{1}{2}\nabla_{(\rho}R^\beta{}_{\mu|\sigma|\nu)}[\frac{S_{11\beta}}{q_0} - \frac{\nabla_\beta S_{11}}{q_0^2}]\Big) \\
\langle k_{0\mu}k_{0\nu}k_{0\rho}k_{0\sigma}y_1^2\rangle &= \frac{1}{4!}\nabla_{(\mu}\nabla_\nu\nabla_\rho\nabla_{\sigma)}\frac{S_{11}}{q_0^2}
\end{aligned} \tag{6.3.33}$$

We now substitute (6.3.33) in the original loop variable expression written in terms of tilde variables (6.3.31). All the higher derivative terms cancel and the final curvature independent term is the expected covariantization of the flat space term: $\nabla_\mu\nabla_\nu S_{11\rho\sigma}$. There are also the terms associated with the naive covariantization. So the final expression starts off as

$$\underbrace{\nabla_\mu\nabla_\nu S_{11\rho\sigma} + \frac{1}{3}(R^\beta{}_{\nu\mu\rho} + R^\beta{}_{\rho\mu\nu})S_{\beta\sigma} + \frac{1}{3}(R^\beta{}_{\nu\mu\sigma} + R^\beta{}_{\sigma\mu\nu})S_{\rho\beta}}_{Naive\ covariantization} + (curvature \times stuckelberg\ fields) \quad (6.3.34)$$

The Stuckelberg fields $S_{11\mu}, S_{11}$ are required for gauge invariance. The gauge transformations are given in (6.2.20). They can be set to zero by a gauge transformation.

The full answer is

$$\begin{aligned}
\langle k_{0\mu}k_{0\nu}k_{1\rho}k_{1\sigma}\rangle \;=\; & \frac{1}{2}[\nabla_{(\mu}\nabla_{\nu)}S_{\rho\sigma} - \frac{1}{3}R^\beta{}_{(\mu|\rho|\nu)}S_{\beta\sigma} - \frac{1}{3}R^\beta{}_{(\mu|\sigma|\nu)}S_{\rho\beta}] + \\
& \frac{1}{6}[R^\beta{}_{(\mu|\rho|\nu)}(\frac{\nabla_{(\beta}S_{\sigma)}}{q_0} - \frac{\nabla_\beta\nabla_\sigma S}{q_0^2}) + R^\beta{}_{(\mu|\sigma|\nu)}(\frac{\nabla_{(\beta}S_{\rho)}}{q_0} - \frac{\nabla_\beta\nabla_\rho S}{q_0^2})] - \\
& \frac{1}{6}[R^\beta{}_{(\mu|\sigma|\nu}\nabla_{\rho)}(\frac{S_\beta}{q_0} - \frac{\nabla_\beta S}{q_0^2}) - \frac{1}{2}\nabla_{(\rho}R^\beta{}_{\mu|\sigma|\nu)}(\frac{S_\beta}{q_0} - \frac{\nabla_\beta S}{q_0^2})] \\
& -\frac{1}{2}[3R^\beta{}_{\sigma\rho\nu}\nabla_\mu S_\beta + 3R^\beta{}_{\sigma\rho\mu}\nabla_\nu S_\beta + R^\beta{}_{\nu\rho\mu}\nabla_\beta S_\sigma + R^\beta{}_{\mu\rho\nu}\nabla_\beta S_\sigma + 3R^\beta{}_{\rho\nu\mu}\nabla_\beta S_\sigma + 3R^\beta{}_{\sigma\nu\mu}\nabla_\rho S_\beta] \\
& +2(\nabla_\nu R^\beta{}_{\sigma\rho\mu})S_\beta + 2(\nabla_\mu R^\beta{}_{\sigma\rho\nu})S_\beta] + \frac{1}{2}[R^\beta{}_{\sigma\nu\mu}\nabla_\rho S_\beta + R^\beta{}_{\rho\nu\mu}\nabla_\beta S_\sigma] \\
\equiv\; & F_{\mu\nu\rho\sigma} \qquad (6.3.35)
\end{aligned}$$

The EOM (6.3.29) can now be written in terms of F as

$$G^{\alpha\beta}[F_{\alpha\beta\mu\nu} - F_{\alpha(\mu|\beta|\nu)} + F_{\mu\nu\alpha\beta}] + q_0^2 S_{\mu\nu} - q_0\nabla_{(\mu}S_{\nu)} + \nabla_\mu\nabla_\nu S = 0 \qquad (6.3.36)$$

where $G_{\mu\nu}$ is the space-time metric. This equation is both manifestly covariant and also gauge invariant by construction. Note that the Stuckelberg fields S_μ, S can be set to zero by a gauge transformation (6.2.20).

6.4 Interacting Equation

The interaction equation can be written in curved space using the same techniques. The main new complication is that the equation involves fields (or more precisely, gauge invariant field strengths) at two different space time points. Thus when we expand the exponential as in (6.1.1) and (6.1.2), it is about the origin of the RNC. For the interacting case we have two such expansions, both about the same origin. The origin has been taken as $z = 0$, with $\bar{Y}(0) = 0$. It is also possible to take one of the points to be $z = 0$. In that case only one of the exponentials have to be expanded.

A typical interaction terms is of the form

$$\int dz_1 dz_2 \dot{G}(z_1, z_2, a) K_{1\mu\nu\rho}[k_n] e^{ik_0.\bar{Y}(z_1)} \bar{Y}_1^\mu \bar{Y}_1^\nu \bar{Y}_1^\rho(z_1) K_{2\alpha\beta\gamma}[k'_n] e^{ik'_0.\bar{Y}(z_2)} \bar{Y}_1^\alpha \bar{Y}_1^\beta \bar{Y}_1^\gamma(z_2) \qquad (6.4.37)$$

As mentioned above, the exponentials have to be understood as a power series that stands for the Taylor expansion described in (A.3.17). In mapping these expressions to space-time fields the same four step procedure can be followed. The point of departure being that each interaction product will involve an infinite series of terms involving higher derivatives

The vertex operators can be taken to be normal ordered because the free equation takes into account all the contributions due to self contractions.

Thus for the OPE of normal ordered exponentials in flat space we have:

$$\begin{aligned}
: e^{ik_0.Y(z_1)} :: e^{ip_0.Y(z_2)} : \;&=\; e^{-k_{0\mu}p_{0\nu}\langle Y^\mu(z_1)Y^\nu(z_2)\rangle} : e^{ik_0.Y(z_1)+ip_0.Y(z_2)} : \\
&=\; e^{-k_{0\mu}p_{0\nu}\langle Y^\mu(z_1)(Y^\nu(z_1)+(z_2-z_1)\partial_z Y^\nu(z_1)+\frac{(z_1-z_2)^2}{2!}\partial_z^2 Y^\nu(z_1)+\ldots\rangle} \\
&\quad : e^{ik_0.Y(z_1)+ip_0.(Y^\nu(z_1)+(z_2-z_1)\partial_z Y^\nu(z_1)+\frac{(z_1-z_2)^2}{2!}\partial_z^2 Y^\nu(z_1)+\ldots)} : \qquad (6.4.38)
\end{aligned}$$

Note that the contraction involves the green function $\langle Y^\mu(z_1)Y^\nu(z_2)\rangle$, which has been Taylor expanded. This can be done in curved space time also, provided we work in RNC. Each term in the expansion of the Green function is a *function* of $z_1 - z_2$ and the cutoff a. The tensor structure in flat space is $\eta^{\mu\nu}$ which becomes $\bar{g}^{\mu\nu}(0)$ in the RNC and $g^{\mu\nu}(x_0)$ in a general coordinate system. The equations obtained are local ones, for fields at x_0, which is any arbitrary point on the manifold - so these equations are completely general.

The expansion of $\bar{Y}(z)$ in the exponent generates higher dimensional vertex operators in flat space. Mapping this RNC expansion to curved space covariantly gives (A.4.25). We see in general that the Riemann tensor appears at higher orders.

One can easily generalize (6.4.38) to do an OPE for arbitrary vertex operators and perform the same Taylor expansion using (A.3.17) and (A.4.25) as done for example in (A.5.31) of the Appendix A (A).

We give a few examples below: (We let X_o stand for the coordinates, in a general coordinate system X, of the point that is the origin for the RNC system. The coordinates in the RNC will be called $\bar{Y}$. The origin is $\bar{Y} = 0$. We also take for the z dependnece $\bar{Y}(0) = 0$.)

Let us take as an example:

$$\langle K_{\alpha\beta\gamma}[p_0, p_n]\rangle = F_{\alpha\beta\gamma}(\bar{Y} = 0) \to F_{\alpha\beta\gamma}(X_o) \tag{6.4.39}$$

Then

$$\begin{aligned}
\langle p_{0\mu} K_{\alpha\beta\gamma}[p_0, p_n]\rangle &= \bar{\nabla}_\mu F_{\alpha\beta\gamma}(\bar{Y} = 0) \to \nabla_\mu F_{\alpha\beta\gamma}(X_o) &(6.4.40)\\
\langle p_{0\mu} p_{0\nu} K_{\alpha\beta\gamma}[p_0, p_n]\rangle &= \bar{\nabla}_\mu \bar{\nabla}_\nu F_{\alpha\beta\gamma}(\bar{Y} = 0) + \frac{1}{3}\Big((\bar{R}^\lambda{}_{\alpha\mu\nu} + \bar{R}^\lambda{}_{\nu\mu\alpha})F_{\lambda\beta\gamma}(0) + \\
&\quad (\bar{R}^\lambda{}_{\beta\mu\nu} + \bar{R}^\lambda{}_{\nu\mu\beta})F_{\alpha\lambda\gamma}(0) + (\bar{R}^\lambda{}_{\gamma\mu\nu} + \bar{R}^\lambda{}_{\nu\mu\gamma})F_{\alpha\beta\lambda}(0)\Big)\\
&\to \nabla_\mu \nabla_\nu F_{\alpha\beta\gamma}(Y_o) + \frac{1}{3}\Big((R^\lambda{}_{\alpha\mu\nu} + R^\lambda{}_{\nu\mu\alpha})F_{\lambda\beta\gamma}(X_0) + \\
&\quad (R^\lambda{}_{\beta\mu\nu} + R^\lambda{}_{\nu\mu\beta})F_{\alpha\lambda\gamma}(X_0) + (R^\lambda{}_{\gamma\mu\nu} + R^\lambda{}_{\nu\mu\gamma})F_{\alpha\beta\lambda}(X_0)\Big) &(6.4.41)
\end{aligned}$$

All these expressions are tensors at the origin X_0 of the RNC. Also for the contractions one needs Taylor expansions of the Green function, for example

$$\langle \bar{Y}^\mu(z_1)\bar{Y}^\nu(z_1)\rangle = \eta^{\mu\nu}G(z_1, z_1; a) \quad ; \quad \langle \bar{Y}^\mu(z_1)\partial_z \bar{Y}^\nu(z_1)\rangle = \eta^{\mu\nu}\partial_{z_2}G(z_1, z_2; a)|_{z_2=z_1} \tag{6.4.42}$$

In a general coordinate system we simply replace $\eta^{\mu\nu}$ by $g^{\mu\nu}(X_0)$ in the above equation.

Putting all this together one obtains on expanding the exponentials in (6.4.38)

$$\begin{gathered}
\langle -\int\int dz_1 dz_2\ [\dot{G}(z_1, z_1; a) + (z_1 - z_2)\dot{G}'(z_1, z_1; a) + ...][1 - k_{0\mu} p_{0\nu} g^{\mu\nu}(Y_0)G(z_1, z_1; a) + ...]\\
K_{\lambda\sigma\rho}[k_0, k_n]\partial_{z_1}Y^\lambda(z_1)\partial_{z_1}Y^\sigma(z_1)\partial_{z_1}Y^\rho(z_1)K_{\alpha\beta\gamma}[p_0, p_n]\partial_{z_1}Y^\alpha(z_1)\partial_{z_1}Y^\beta(z_1)\partial_{z_1}Y^\gamma(z_1)\\
e^{i(k_0+p_0).Y_0}(1 + (z_2 - z_1)p_0.\partial_{z_1}Y(z_1) + ...)\rangle\\
= -\int\int dz_1 dz_2\ [\dot{G}(z_1, z_1; a) + (z_2 - z_1)\dot{G}'(z_1, z_1; a) + ...]\\
[F_{\lambda\sigma\rho}(X_0)F_{\alpha\beta\gamma}(X_0) - g^{\mu\nu}(X_0)\nabla_\mu F_{\lambda\sigma\rho}(X_0)\nabla_\nu F_{\alpha\beta\gamma}(X_0)G(z_1, z_1; a) + ...]\\
\partial_{z_1}Y^\lambda(z_1)\partial_{z_1}Y^\sigma(z_1)\partial_{z_1}Y^\rho(z_1)\partial_{z_1}Y^\alpha(z_1)\partial_{z_1}Y^\beta(z_1)\partial_{z_1}Y^\gamma(z_1) + ...
\end{gathered} \tag{6.4.43}$$

where we have kept a sample term at level 6. This generates terms at lower levels from contractions as in:

$$\begin{gathered}
\partial_{z_1}Y^\lambda(z_1)\partial_{z_1}Y^\sigma(z_1)\partial_{z_1}Y^\rho(z_1)\partial_{z_1}Y^\alpha(z_1)\partial_{z_1}Y^\beta(z_1)\partial_{z_1}Y^\gamma(z_1) =\\
: \partial_{z_1}Y^\lambda(z_1)\partial_{z_1}Y^\sigma(z_1)\partial_{z_1}Y^\rho(z_1)\partial_{z_1}Y^\alpha(z_1)\partial_{z_1}Y^\beta(z_1)\partial_{z_1}Y^\gamma(z_1) : +\\
\partial_{z_1}\partial_{z_2}\langle Y^\lambda(z_1)Y^\alpha(z_2)\rangle|_{z_1=z_2} : \partial_{z_1}Y^\sigma(z_1)\partial_{z_1}Y^\rho(z_1)\partial_{z_1}Y^\beta(z_1)\partial_{z_1}Y^\gamma(z_1) : +...
\end{gathered}$$

and in addition there are also terms at higher levels coming from expanding the exponential.

7 Closed String Theory

In [42] gauge invariant free equations for closed string was written down. The open string loop variable is extended to closed strings by adding the anti-holomorphic counterparts. Thus the starting point is

$$e^{ik_0.X(z)+\oint_c dt\ k(t)\alpha(t)\partial_z X(z+t)+\oint_c d\bar{t}\ \bar{k}(\bar{t})\bar{\alpha}(\bar{t})\partial_{\bar{z}}X(\bar{z}+\bar{t})} \tag{7.0.1}$$

Expanding, we get

$$\int dz\ e^{i(k_0.Y+\sum_{n,\bar{n}=1,2,...}(k_n.Y_n+k_{\bar{n}}.Y_{\bar{n}}))} = \int dz\ \Big(e^{ik_0Y}(1+ik_n.Y_n+ik_{\bar{n}}.Y_{\bar{n}}-k_{n\mu}k_{\bar{m}\nu}Y_n^\mu Y_{\bar{m}}^\nu+...\Big) \tag{7.0.2}$$

The holomorphic anti-holomorphic separation originates in the world sheet equation of motion for $X(z,\bar{z})$

$$\partial_z\partial_{\bar{z}}X=0$$

This means that physical states are represented by vertex operators that are products of terms of the form $\partial_z^n X$ and $\partial_{\bar{z}}^n X$ but do not involve mixed derivatives $\partial_z\partial_{\bar{z}}X$. The constraint $L_0=\bar{L}_0$ sets the dimensions of the holomorphic and anti holomorphic parts equal. In [42] Weyl invariance was used to obtain gauge invariant equations. The method is a simple extension of what is described in Section 3 of this review. One noteworthy feature of the method is that terms involving mixed derivatives $\partial_z\partial_{\bar{z}}\Sigma$ have to be retained. When the Liouville field is varied, on integration by parts, they contribute terms to the equation of motion. One way to understand the necessity of adding mixed derivative terms is the following [33]. While the unregulated Green function $G(z,\bar{z};0)=ln\ (z\bar{z})$ obeys $\partial_z\partial_{\bar{z}}G(z,\bar{z};0)=0$, the regulated Green function, which can be taken to be

$$G(z,\bar{z};0)=ln\ (z\bar{z}+a^2e^{2\sigma})$$

does not. Thus as long as there is a cutoff in place, holomorphic factorization does not take place. Therefore one should introduce vertex operators involving mixed derivatives. In the limit that $a\to 0$, which can be taken for on-shell S-matrix element calculation, vertex operators that correspond to mixed derivatives will have zero correlators with other vertex operators. These states therefore will not contribute to the S-matrix. But for off shell calculations one should retain these states.

There is also another argument for the introduction of mixed derivatives. Recall the argument for gauge invariance of the interacting term. It was observed that the gauge variation of the Lagrangian at level N has to be derivatives of lower level terms in the Lagrangian. For closed strings we have λ_n as well as $\lambda_{\bar{n}}$. Thus if L_N denotes the Lagrangian at level N its gauge variation has to be of the form:

$$\delta L_N=\sum_{n,\bar{n}=1,2,...}\lambda_n\frac{\partial L_{N-n}}{\partial x_n}+\lambda_{\bar{n}}\frac{\partial L_{N-\bar{n}}}{\partial \bar{x}_n} \tag{7.0.3}$$

where for closed strings, the level involves two numbers: $N=n+\bar{m}$. Since $L_{N-\bar{n}}$ certainly has purely holomorphic derivative terms (and L_{N-n} certainly has purely anti holomorphic derivative terms), it is clear that L_N must have mixed derivative terms.

Motivated by these considerations we generalize our loop variable [33] to

$$Exp\ \Big(i\Big(k_0.X(z)+\oint_c dt\ k(t)\alpha(t)\partial_z X(z+t)+\oint_c d\bar{t}\ \bar{k}(\bar{t})\bar{\alpha}(\bar{t})\partial_{\bar{z}}X(\bar{z}+\bar{t})+$$

$$+\oint_c dt\oint_c d\bar{t}\ K(t,\bar{t})\alpha(t)\bar{\alpha}(\bar{t})\partial_z\partial_{\bar{z}}X(z+t,\bar{z}+\bar{t})\Big)\Big) \tag{7.0.4}$$

Expansion for $k(t),\alpha(t)$ are as given earlier and $\bar{k}(\bar{t}),\bar{\alpha}(\bar{t})$ are anti-holomorphic versions of the same. The first three terms in the exponent are the terms given in (7.0.2). The fourth term involves $K(t,\bar{t})$ defined below:

$$K(t,\bar{t})\equiv K_{0;0}+\sum_{\bar{m}=1}^{\infty}K_{0;\bar{m}}\bar{t}^{-\bar{m}}+\sum_{n=1}^{\infty}K_{n;0}t^{-n}+\sum_{n=1,\bar{m}=1}^{\infty}K_{n;\bar{m}}t^{-n}\bar{t}^{-\bar{m}} \tag{7.0.5}$$

Expanding $X(z+t,\bar{z}+\bar{t})$ gives

$$\partial_z\partial_{\bar{z}}X(z+t,\bar{z}+\bar{t}) = \partial_z\partial_{\bar{z}}X + t\partial_z^2\partial_{\bar{z}}X + \bar{t}\partial_z\partial_{\bar{z}}^2X + t\bar{t}\partial_z^2\partial_{\bar{z}}^2X + t^2\frac{\partial_z^3\partial_{\bar{z}}X}{2!} + t^2\frac{\partial_z\partial_{\bar{z}}^3X}{2!} + ... \tag{7.0.6}$$

Plugging all this in (7.0.4) gives:

$$k_0\Big(X + \alpha_1\partial_zX + \alpha_2\partial_z^2X + \frac{\alpha_3\partial_z^3X}{2!} + ... + \bar{\alpha}_1\partial_zX + \bar{\alpha}_2\partial_{\bar{z}}^2X + ... + \frac{\alpha_n\bar{\alpha}_m\partial_z^n\partial_{\bar{z}}^mX}{(n-1)!(m-1)!} + ..\Big)$$

$$+\underbrace{K_{1;0}}_{=k_1}\Big(\partial_zX + \alpha_1\partial_z^2X + \frac{\alpha_2\partial_z^3X}{2!} + .. + \bar{\alpha}_1\partial_z\partial_{\bar{z}}X + \bar{\alpha}_2\partial_z\partial_{\bar{z}}^2X +$$

$$... + \alpha_1\bar{\alpha}_1\partial_z^2\partial_{\bar{z}}X + \frac{\alpha_2\bar{\alpha}_1\partial_z^3\partial_{\bar{z}}X}{2!} + ... + \alpha_1\bar{\alpha}_2\partial_z^2\partial_{\bar{z}}^2X + ... + \frac{\alpha_n\bar{\alpha}_m\partial_z^{n+1}\partial_{\bar{z}}^mX}{n!(m-1)!} + ...\Big) +$$

$$... + K_{n;\bar{m}}\Big(\frac{\partial_z^n\partial_{\bar{z}}^mX}{(n-1)!(m-1)!} + \frac{\alpha_1\partial_z^{n+1}\partial_{\bar{z}}^mX}{(n)!(m-1)!} + ... + \frac{\alpha_p\bar{\alpha}_q\partial_z^{n+p}\partial_{\bar{z}}^{m+q}X}{(n+p-1)!(m+q-1)!} + ...\Big)$$

If we define the coefficient of k_0 to be Y, (7.0.4) can be compactly written as

$$Exp\Big(i\Big(k_0.Y + K_{1;0}.\frac{\partial Y}{\partial x_1} + K_{0;\bar{1}}.\frac{\partial Y}{\partial\bar{x}_1} + K_{1;\bar{1}}.\frac{\partial^2Y}{\partial x_1\partial\bar{x}_1} + ... + K_{n;\bar{m}}.\frac{\partial^2Y}{\partial x_n\partial\bar{x}_m} + ...\Big)\Big) \tag{7.0.7}$$

This is the generalization to closed strings of $e^{i\sum_n k_nY_n}$. This is correct for the free theory. For the interacting theory we need a finer resolution of the $K's$. For instance although $\frac{\partial^2Y}{\partial x_nx_m\bar{x}_n\bar{x}_m} = \frac{\partial^2Y}{\partial x_{n+m}\partial x_{\bar{n}+\bar{m}}}$ we need separate terms for each. Thus $K_{n,\bar{m}}$ has to be generalized to $K_{[n],[\bar{m}]}$ where $[n],[\bar{m}]$ denote all the partitions of $n,\bar{m}$ respectively. In the next section we give the construction of $K_{[n],[\bar{m}]}$.

7.1 Construction of $K_{[n],[\bar{m}]}$

The starting point for closed strings, is to make the identification

$$K_{\mu n,m...;0} = K_{\mu n,m,..}$$

where the RHS are the K's that we have just defined (for open strings). Similarly $K_{\mu 0;\bar{n},\bar{m}...}$ is given by the same expressions with bars i.e. $\bar{n}$ instead of n, $k_{\bar{1}\mu}$ instead of $k_{1\mu}$ etc.

Now let us construct the mixed K's:

$$K_{1;\bar{1}\mu} = \bar{y}_1k_{1\mu} + y_1k_{\bar{1}\mu} - y_1\bar{y}_1k_{0\mu} = \bar{y}_1K_{1;0\mu} + y_1K_{0;\bar{1}\mu} - y_1\bar{y}_1k_{0\mu} \tag{7.1.8}$$

It is easily verified that

$$\delta K_{1;\bar{1}\mu} = \lambda_1k_{\bar{1}\mu} + \bar{\lambda}_1k_{1\mu}$$

Similarly

$$K_{1,1;\bar{1}\mu} = \bar{y}_1(k_{2\mu} - y_2k_{0\mu}) + \frac{y_1^2}{2}k_{\bar{1}\mu} - \frac{y_1^2}{2}\bar{y}_1k_{0\mu}$$

This can be rewritten as

$$K_{1,1;\bar{1}\mu} = \bar{y}_1K_{1,1;0\mu} + \frac{y_1^2}{2}K_{0;\bar{1}\mu} - \frac{y_1^2}{2}\bar{y}_1k_{0\mu} \tag{7.1.9}$$

One can easily check that

$$\delta K_{1,1;\bar{1}\mu} = \lambda_1K_{1;\bar{1}\mu} + \bar{\lambda}_1K_{1,1;0\mu}$$

as required. The pattern is very clear:

$$K_{\underbrace{1,1,...1}_{n};\underbrace{\bar{1},\bar{1},...\bar{1}}_{m}\mu} = \frac{\bar{y}_1^m}{m!}K_{\underbrace{1,1,...,1}_{n};0\mu} + \frac{y_1^n}{n!}K_{0;\underbrace{\bar{1},\bar{1},...,\bar{1}}_{m}\mu} - \frac{y_1^n}{n!}\frac{\bar{y}_1^m}{m!}k_{0\mu} \tag{7.1.10}$$

Let us check the variation:

$$\begin{aligned}
\delta K_{\underbrace{1,1,...1}_{n};\underbrace{\bar{1},\bar{1},...\bar{1}}_{m}\mu} &= \bar{\lambda}_1\Big(\frac{\bar{y}_1^{m-1}}{(m-1)!}K_{\underbrace{1,1,...,1}_{n};0\mu} + \frac{y_1^n}{n!}K_{0;\underbrace{\bar{1},\bar{1},...,\bar{1}}_{m-1}\mu} - \frac{y_1^n}{n!}\frac{\bar{y}_1^{m-1}}{(m-1)!}k_{0\mu}\Big) \\
&+ \lambda_1\Big(\frac{\bar{y}_1^m}{m!}K_{\underbrace{1,1,...,1}_{n-1};0\mu} + \frac{y_1^{n-1}}{(n-1)!}K_{0;\underbrace{\bar{1},\bar{1},...,\bar{1}}_{m}\mu} - \frac{y_1^{n-1}}{(n-1)!}\frac{\bar{y}_1^m}{m!}k_{0\mu}\Big) \\
&= \bar{\lambda}_1 K_{\underbrace{1,1,...1}_{n};\underbrace{\bar{1},\bar{1},...\bar{1}}_{m-1}\mu} + \lambda_1 K_{\underbrace{1,1,...1}_{n-1};\underbrace{\bar{1},\bar{1},...\bar{1}}_{m}\mu}
\end{aligned} \tag{7.1.11}$$

One can then see that

$$K_{p_1,p_2,..,\underbrace{1,1,...1}_{n};\bar{q}_1,\bar{q}_2,..,\underbrace{\bar{1},\bar{1},...\bar{1}}_{m}\mu} = y_{p_1}y_{p_2}...\bar{y}_{q_1}\bar{y}_{q_2}..K_{\underbrace{1,1,...1}_{n};\underbrace{\bar{1},\bar{1},...\bar{1}}_{m}\mu} \tag{7.1.12}$$

with

$$p_1, p_2, ..., q_1, q_2, ; \geq 2, \quad p_1 \neq p_2 \neq ...; \bar{q}_1 \neq \bar{q}_2 \neq ...$$

has the right gauge transformation. If any of the p are repeated i times, then y_p is replaced by $\frac{y_p^i}{i!}$. Similarly for the $\bar{y}_q$.

This completes the construction of $K_{\mu[n];[\bar{m}]}$. Since the basic variables are $k_n, k_{\bar{n}}, q_n, q_{\bar{n}}$ it is clear that no new degrees of freedom have been added to that of the free theory. However, in principle one could add to $K_{[n]_i;[\bar{m}]_j\mu}$, new variables of the form $k_{[n]_i;[\bar{m}]_j\mu}$ with transformation rule

$$\delta k_{[n]_i;[\bar{m}]_j\mu} = \lambda_p k_{[n]_i/p;[\bar{m}]_j\mu} + \bar{\lambda}_p k_{[n]_i;[\bar{m}]/\bar{p}\mu} \tag{7.1.13}$$

where as earlier $[n]_i/p$ stands for the particular partition $[n]_i$ with the one p removed. (If $[n]_i$ does not contain p, that term does not contribute to the gauge transformation, and can be set to zero.) This is also discussed in Appendix D (D).

7.1.1 An interesting relation

The K's obey a relation of the form:

$$\tilde{K}_{n;\bar{m}\mu} \equiv \sum_{i,j} K_{[n]_i;[\bar{m}]_j\mu} = \bar{q}_n k_{\bar{m}\mu} + \bar{q}_{\bar{m}} k_{n\mu} - \bar{q}_n \bar{q}_{\bar{m}} k_{0\mu} \tag{7.1.14}$$

Here, as earlier $[n]_i$ denotes a particular partition of n denoted by i and $\bar{q}$ was defined in (5.4.38). Thus for instance

$$\tilde{K}_{2,\bar{1}\mu} \equiv K_{2;\bar{1}\mu} + K_{1,1;\bar{1}\mu} = \bar{q}_2 k_{\bar{1}\mu} + \bar{q}_{\bar{1}} k_{2\mu} - \bar{q}_{\bar{1}}\bar{q}_2 k_{0\mu}$$

The gauge transformation of $\tilde{K}_{n;\bar{m}\mu}$ under λ_p is easily seen to be:

$$\delta\tilde{K}_{n;\bar{m}\mu} = \lambda_p \tilde{K}_{n-p;\bar{m}\mu} \tag{7.1.15}$$

Proof: The only partitions $[n]_i$ that contribute to the gauge transformation, are the ones that have at least one p. Take these partitions and remove one p. The remaining numbers are all possible ways of making $n-p$

- so we get all the partitions of $n-p$. The gauge transformation law then forces (7.1.14) to be true. This relation will be used in the construction of the free equations.

For the free equations one has to keep only single derivatives in the loop variable. Thus we write $\frac{\partial}{\partial x_{n_1+n_2+..}}\frac{\partial}{\partial x_{\bar{m}_1+\bar{m}_2+..}}Y$ for $\partial_{n_1}\partial_{n_2}...\partial_{\bar{m}_1}\partial_{\bar{m}_2}Y....$ Thus the coefficient of $Y^{\mu}_{n_1+n_2+...;\bar{m}_1+\bar{m}_2+..}$ is $\tilde{K}_{n;\bar{m}\mu}$

Of course one can still add some new variables $k_{[n]_i,[\bar{m}]_j\mu}$ with the correct gauge transformation law (7.1.13), as mentioned earlier and then this would contribute to $\tilde{K}_{n,\bar{m}\mu}$ also. This is in fact done in Appendix C (D).

7.2 Free Equation

As in the case of the open string, equations are given by the ERG (5.3.21).

$$\int du\ \frac{\partial L}{\partial \tau}\psi = \int dzdz'\ \{-\frac{1}{2}\underbrace{\dot{G}(z,z')G^{-1}(z,z')}_{field\ independent} -$$

$$\frac{1}{2}\dot{G}(z,z')[\frac{\delta^2}{\delta X(z)\delta X(z')}\int du\ L[X(u),X'(u)]] + \frac{\delta}{\delta X(z)}\int du\ L[X(u)]\frac{\delta}{\delta X(z')}\int du'\ L[X(u')]\}\psi \qquad (7.2.16)$$

The second term, when $L[Y(u)]$ is taken to be the loop variable (5.4.22) was worked out in Sec 5.4. We need to generalize this to the loop variable for closed strings (7.0.7). This is straightforward - we need to include derivatives w.r.t. the anti holomorphic variables $Y_{\bar{n}} \equiv \frac{\partial Y}{\partial \bar{x}_n}$ and also mixed derivatives $Y_{n,\bar{m}} \equiv \frac{\partial Y}{\partial x_n \partial \bar{x}_m}$.

Thus (5.4.23) is generalized to

$$\begin{aligned}\int du\ \frac{\delta}{\delta Y^{\mu}(z')}\mathcal{L}(u) \quad &= \quad \int du\ \Big\{\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}(u)}\delta(u-z') + \frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_1(u)}\partial_{x_1}\delta(u-z') \\ &+ \quad \frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{\bar{1}}(u)}\partial_{\bar{x}_1}\delta(u-z') + \frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{1;\bar{1}}(u)}\partial_{x_1}\partial_{\bar{x}_1}\delta(u-z') \\ &+ \quad \frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_2(u)}\partial_{x_2}\delta(u-z') + \frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{\bar{2}}(u)}\partial_{\bar{x}_2}\delta(u-z')\end{aligned}$$

$$+\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{2;\bar{1}}(u)}\partial_{x_2}\partial_{\bar{x}_1}\delta(u-z')+...+\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_n(u)}\partial_{x_n}\delta(u-z')+..\Big\} \qquad (7.2.17)$$

The variables $x_n, \bar{x}_m$ in the derivatives acting on delta functions, correspond to u, and we can integrate by parts on u. Thus for the second derivative $\frac{\delta^2}{\delta X^{\mu}(z')\delta X_{\mu}(z'')}$ we get

$$\eta^{\mu\nu}\frac{\delta}{\delta X^{\nu}(z')}\int du\ \Big\{\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}(u)}\delta(u-z'') - \partial_{x_1}\frac{\partial \mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^{\mu}_{1;0}(u)}\delta(u-z'')$$

$$-\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^{\mu}_{0;\bar{1}}(u)}\delta(u-z'') + \partial_{x_1}\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{1,\bar{1}}(u)}\delta(u-z'') + ...\Big\} \qquad (7.2.18)$$

$$= \eta^{\mu\nu}\int du\ [\frac{\partial}{\partial Y^{\nu}(u)} + \frac{\partial}{\partial x_1}\delta(u-z')\frac{\partial}{\partial Y^{\nu}_{1;0}(u)} + \frac{\partial}{\partial \bar{x}_1}\delta(u-z')\frac{\partial}{\partial Y^{\nu}_{0;\bar{1}}(u)} + \frac{\partial}{\partial x_2}\delta(u-z')\frac{\partial}{\partial Y^{\nu}_{2;0}(u)} +$$

$$\frac{\partial}{\partial \bar{x}_2}\delta(u-z')\frac{\partial}{\partial Y^{\nu}_{0;\bar{2}}(u)} + \frac{\partial^2}{\partial x_1 \partial \bar{x}_1}\delta(u-z')\frac{\partial}{\partial Y^{\nu}_{1;\bar{1}}(u)} + ...]$$

$$\Big\{\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}(u)}\delta(u-z'') - \partial_{x_1}\frac{\partial \mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^{\mu}_1(u)}\delta(u-z'')$$

$$-\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^{\mu}_{\bar{1}}(u)}\delta(u-z'') + \partial_{x_1}\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u),Y_{n,\bar{m}}(u)]}{\partial Y^{\mu}_{1,\bar{1}}(u)}\delta(u-z'') + ...\Big\} \qquad (7.2.19)$$

The details of the algebra are worked out in Appendix B (B). The free equation was derived using Weyl invariance in [42]. One difference between that derivation and this one is noteworthy. There it was assumed that $\partial_z \partial_{\bar{z}} X(z,\bar{z}) = 0$. So there were no mixed derivative vertex operators. However $\partial_z \partial_{\bar{z}} \sigma \neq 0$. Thus in the loop variable generalization, this meant that $\partial_{x_n} \partial_{\bar{x}_m} \Sigma \neq 0$. Thus, if we assume that mixed derivatives are absent, then in terms of Green functions

$$\partial_{x_n} \partial_{\bar{x}_m} \Sigma = \partial_{x_n} \partial_{\bar{x}_m} \langle Y(z,\bar{z}) Y(z,\bar{z}) \rangle = 2 \langle \partial_{x_n} Y(z,\bar{z}) \partial_{\bar{x}_m} Y(z,\bar{z}) \rangle \tag{7.2.20}$$

On the other hand in the present approach

$$\partial_{x_n} \partial_{\bar{x}_m} \Sigma = \partial_{x_n} \partial_{\bar{x}_m} \langle Y(z,\bar{z}) Y(z,\bar{z}) \rangle = 2 \langle \partial_{x_n} Y(z,\bar{z}) \partial_{\bar{x}_m} Y(z,\bar{z}) \rangle + 2 \langle \partial_{x_n} \partial_{\bar{x}_m} Y(z,\bar{z}) Y(z,\bar{z}) \rangle \tag{7.2.21}$$

In the Weyl invariance approach of [42] the coefficient of the term in (7.2.20) is $k_n . k_{\bar{m}}$. In the present approach the two terms on the RHS of (7.2.21) come with *a priori* different coefficients: the first with $k_n . k_{\bar{m}}$ and the second with $K_{n,\bar{m}} . k_0$. Thus when we set these two equal, we get the same (correct) free equations. These constraints were called K-constraints in [33] and are derived in Appendix D (D).

We first derive the EOM in flat space time. Later we generalize to curved space time.

7.2.1 Results: Graviton

We get for level $(1, \bar{1})$:

$$[-k_0^2 k_{1\mu} k_{\bar{1}\nu} + k_0 . k_1 k_{0\mu} k_{\bar{1}\nu} + k_0 . k_{\bar{1}} k_{0\nu} k_{1\mu} - k_1 . k_{\bar{1}} k_{0\mu} k_{0\nu}] Y^{\mu}_{1;0} Y^{\nu}_{0;\bar{1}} = 0 \tag{7.2.22}$$

Let us write this equation in terms of space time fields and analyze the gauge transformations: Define

$$\langle \frac{1}{2} k_{1(\mu} k_{\bar{1}\nu)} \rangle = h_{\mu\nu}$$

$$\langle \frac{1}{2} k_{1[\mu} k_{\bar{1}\nu]} \rangle = B_{\mu\nu} \tag{7.2.23}$$

Let us also define

$$\langle \frac{1}{2} (\lambda_1 k_{\bar{1}\mu} + \bar{\lambda}_1 k_{1\nu}) \rangle = \epsilon_\mu$$

$$\langle \frac{1}{2} (\lambda_1 k_{\bar{1}\mu} - \bar{\lambda}_1 k_{1\nu}) \rangle = \Lambda_\mu \tag{7.2.24}$$

Then the gauge transformation laws are

$$\delta_G h_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)}$$

$$\delta_G B_{\mu\nu} = \partial_{[\mu} \Lambda_{\nu]} \tag{7.2.25}$$

The equation splits into two parts:

$$-\Box h_{\mu\nu} + \partial^\rho \partial_\mu h_{\rho\nu} + \partial^\rho \partial_\nu h_{\mu\rho} - \partial_\mu \partial_\nu h^\rho{}_\rho = 0 \tag{7.2.26}$$

the linearized graviton equation about flat space time, and

$$\partial^\rho [\partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}] = 0 \tag{7.2.27}$$

The quantity in square brackets is just the field strength ($H = dB$) for B:

$$H_{\rho\mu\nu} = \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}$$

The graviton equation can also be written as

$$2\partial^\rho \Gamma_{\rho\mu\nu} - \partial_\mu \partial_\nu h^\rho{}_\rho = 0 \tag{7.2.28}$$

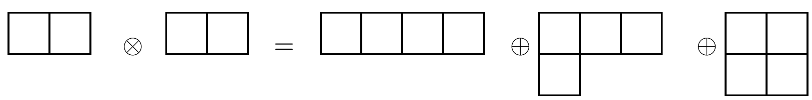
where $\Gamma_{\rho\mu\nu}$ is the Christoffel connection

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}[\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}]$$

Note that at the linearized level ϵ_μ does not have the interpretation of a coordinate transformation. At this level it is a gauge parameter with no geometrical interpretation. When we include interactions we will be forced to the geometrical interpretation. It is then tempting to speculate that the gauge parameter for the B field also has such an interpretation. In that case space time would seem to be complex with ϵ_μ and Λ_μ being the transformation of the real and imaginary parts respectively [33].

7.2.2 Level $(2, \bar{2})$: Spin 4

1. **Physical States** The closed string physical states are direct products of the open string states. We have seen that for open strings the states at level 2 come from a two index traceless symmetric tensor. But the covariant description requires the trace. Thus we have the diagram



The gauge invariant description requires many other tensor fields. In open strings q_1 was not allowed and had to be replaced. The corresponding rule for closed strings was discussed above, is that the number of q_1's and $q_{\bar{1}}$'s should be equal.

2. **Field Content and Gauge Transformation**

- **Scalars** The allowed combinations are:

$$\delta S^{2\bar{2}} = \langle \delta(q_2 q_{\bar{2}}) \rangle = \langle 2\lambda_2 q_0 q_{\bar{2}} + 2\lambda_{\bar{2}} q_{\bar{0}} q_2 \rangle = 2\Lambda^{2\bar{2}} q_0 + 2\Lambda^{\bar{2}2} q_{\bar{0}} \tag{7.2.29}$$

Since ${q_1}^2 {\bar{q_1}}^2 = q_2 q_{\bar{2}} q_0 q_{\bar{0}}$, $S^{11\bar{1}\bar{1}}$ is not an independent field. Here we have used the q-rules separately for the left and right modes separately:

$$q_1^2 = q_2 q_0; \quad \lambda_1 q_1 = \lambda_2 q_0 \tag{7.2.30}$$

- **Vectors**

$$\begin{aligned} \delta S^{2\bar{2}}_\mu = \langle \delta(k_{2\mu} \bar{q_2}) \rangle &= \langle 2\lambda_{\bar{2}} q_{\bar{0}} k_{2\mu} + \lambda_1 k_{1\mu} \bar{q_2} + k_{0\mu} \lambda_2 \bar{q_2} \rangle = 2 q_{\bar{0}} \Lambda^{\bar{2}2}_{\ \ \mu} + k_{0\mu} \Lambda^{2\bar{2}} \\ \delta S^{\bar{2}2}_\rho = \langle \delta(k_{\bar{2}\rho} q_2) \rangle &= \langle 2\lambda_2 q_0 k_{\bar{2}\rho} + \lambda_{\bar{1}} k_{\bar{1}\rho} q_2 + k_{0\rho} \lambda_{\bar{2}} q_2 \rangle = 2 q_0 \Lambda^{2\bar{2}}_{\ \ \rho} + k_{0\rho} \Lambda^{\bar{2}2} \end{aligned} \tag{7.2.31}$$

We have used a notation that the first number superscript refers to the level of the k's and the second to the q's. For the gauge parameters, the first index refers to the level of λ, the second to k and the third to q. The space time index is directly below the corresponding k-level number.

We have two vectors and four vector gauge parameters. We can thus set

$$\langle \lambda_1 k_{1\mu} \bar{q_2} \rangle = 0 \tag{7.2.32}$$

without any damage to our ability to gauge away Stuckelberg fields.

- **2-Tensors**
  Q-rules relate $q_1\bar{q}_{\bar{1}}k_{1\mu}k_{\bar{1}\rho}$ to $k_{2\mu}k_{\bar{2}\rho}$ so it is not an independent field. Thus we have

$$\begin{aligned}
\delta S^{2\bar{2}}_{\mu\rho} = \langle\delta(k_{2\mu}k_{\bar{2}\rho})\rangle &= \langle\lambda_1 k_{1\mu}k_{\bar{2}\rho} + \lambda_{\bar{1}}k_{\bar{1}\rho}k_{2\mu} + k_{0\mu}\lambda_2 k_{\bar{2}\rho} + k_{0\rho}\lambda_{\bar{2}}k_{2\mu}\rangle = \Lambda^{11\bar{2}}_{\ \ \mu\rho} + \Lambda^{\bar{1}\bar{1}2}_{\ \ \rho\mu} + k_{0\mu}\Lambda^{2\bar{2}}_{\ \ \rho} + k_{0\rho}\Lambda^{\bar{2}2}_{\ \ \mu} \\
\delta S^{11\bar{2}}_{\mu\nu} = \langle\delta(k_{1\mu}k_{1\nu}\bar{q}_{\bar{2}})\rangle &= 2\langle\lambda_{\bar{2}}q_{\bar{0}}k_{1\mu}k_{1\nu} + \underbrace{k_{0(\mu}\lambda_1 k_{1\nu)}\bar{q}_{\bar{2}}}_{=0}\rangle = 2\Lambda^{\bar{2}11}_{\ \ \mu\nu} \\
\delta S^{\bar{1}\bar{1}2}_{\rho\sigma} = \langle\delta(q_2 k_{\bar{1}\rho}k_{\bar{1}\sigma})\rangle &= \langle 2\lambda_2 q_0 k_{\bar{1}\rho}k_{\bar{1}\sigma} + \underbrace{k_{0(\rho}\lambda_{\bar{1}}k_{1\bar{\sigma})}q_2}_{=0}\rangle = 2\Lambda^{2\bar{1}\bar{1}}_{\ \ \rho\sigma}
\end{aligned}$$

  Some of the gauge parameters that have already been set to zero earlier are shown here as being set to zero also.
- **3-Tensor**

$$\delta S^{11\bar{2}}_{\mu\nu\rho} = \langle\delta(k_{1\mu}k_{1\nu}k_{\bar{2}\rho})\rangle = \langle\lambda_{\bar{1}}k_{\bar{1}\rho}k_{1\mu}k_{1\nu} + k_{0(\mu}\lambda_1 k_{1\nu)}k_{\bar{2}\rho} + k_{0\rho}\lambda_{\bar{2}}k_{1\mu}k_{1\nu}\rangle = \Lambda^{\bar{1}\bar{1}11}_{\ \ \rho\mu\nu} + k_{0(\mu}\Lambda^{11\bar{2}}_{\ \ \nu)\rho} + k_{0\rho}\Lambda^{\bar{2}11}_{\ \ \mu\nu} \tag{7.2.33}$$

  There is also the conjugate with bars exchanged.
- **4-Tensor**

We focus on the four index tensor, which contains all the physical states. The four index tensor is also interesting because it includes as shown above, tensors with mixed symmetry.

The world sheet action has a term $(k_1.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2$ corresponding to the 4-tensor:

$$\langle k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle = S^{11\bar{1}\bar{1}}_{\mu\nu\rho\sigma} \tag{7.2.34}$$

We can define tensor irreps by writing (brackets denote symmetrization: $S_{(\mu\sigma)} = S_{\mu\sigma} + S_{\sigma\mu}$) the "resolution of unity":

$$S_{\mu\nu\rho\sigma} = \frac{1}{24}\underbrace{S^{S}_{\mu\nu\rho\sigma}} + \frac{1}{8}\underbrace{S^{(3,1)}_{\mu\nu(\rho\sigma)}} + \frac{1}{12}\underbrace{S^{(2,2)}_{\mu\nu\rho\sigma}} \tag{7.2.35}$$

Hereafter, for simplicity we write $S_{\mu\nu\rho\sigma}$ instead of $S^{11\bar{1}\bar{1}}_{\mu\nu\rho\sigma}$. Its gauge variation is

$$\begin{aligned}
\delta\langle k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle &= \langle\lambda_1 k_{0\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle + \langle\lambda_1 k_{0\nu}k_{1\mu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle + \langle\lambda_{\bar{1}}k_{0\rho}k_{1\mu}k_{1\nu}k_{\bar{1}\sigma}\rangle + \langle\lambda_{\bar{1}}k_{0\sigma}k_{1\mu}k_{1\nu}k_{\bar{1}\rho}\rangle \\
\Longrightarrow\ \delta S_{\mu\nu\rho\sigma} &= \partial_\mu\Lambda^{11\bar{1}\bar{1}}_{\ \ \nu\rho\sigma} + \partial_\nu\Lambda^{11\bar{1}\bar{1}}_{\ \ \mu\rho\sigma} + \partial_\rho\bar{\Lambda}^{\bar{1}\bar{1}11}_{\ \ \sigma\mu\nu} + \partial_\sigma\bar{\Lambda}^{\bar{1}\bar{1}11}_{\ \ \rho\mu\nu}
\end{aligned} \tag{7.2.36}$$

For the gauge transformation parameter $\Lambda_{\nu\rho\sigma} = \langle\lambda_1 k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle$ irreps are defined by the resolution of unity which reads as:

$$\Lambda_{\nu\rho\sigma} = \frac{1}{6}\underbrace{\Lambda^{S}_{\nu\rho\sigma}} - \frac{1}{3}\underbrace{\Lambda^{I}_{\sigma\rho\nu}} \tag{7.2.37}$$

In terms of these fields and gauge parameters one obtains:

$$
\begin{aligned}
\frac{1}{24}\delta S^S_{i_1i_2i_3i_4} &= \frac{1}{12}[\partial_{i_1}\Lambda^S_{i_3i_4i_2}+\partial_{i_2}\Lambda^S_{i_3i_4i_1}+\partial_{i_3}\Lambda^S_{i_4i_1i_2}+\partial_{i_4}\Lambda^S_{i_3i_1i_2}]\\
\frac{1}{8}\delta S^{(3,1)}_{i_1i_2(i_3i_4)} &= \frac{1}{12}[\partial_{(i_1}\Lambda^S_{|i_3i_4|i_2)}-\partial_{(i_3}\Lambda^S_{|i_1i_2|i_4)}]-\frac{1}{6}[\partial_{(i_1}\Lambda^I_{|i_3i_4|i_2)}-\partial_{(i_3}\Lambda^I_{|i_1i_2|i_4)}]\\
\frac{1}{12}\delta S^{(2,2)}_{i_1i_2i_3i_4} &= -\frac{1}{6}[\partial_{(i_1}\Lambda^I_{|i_3i_4|i_2)}+\partial_{(i_3}\Lambda^I_{|i_1i_2|i_4)}]
\end{aligned}
\tag{7.2.38}
$$

There is an identical complex conjugate equation involving $\bar{\Lambda}$ which we do not bother to write down.

3. **Free Equation**

The free equation of motion (EOM) can be written as:

$$
-\frac{1}{4}k_0^2(k_1.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_1(k_0.Y_1)(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_{\bar{1}}(k_0.Y_{\bar{1}})(k_{\bar{1}}.Y_{\bar{1}})(k_1.Y_1)^2+
$$

$$
-\frac{1}{4}k_1.k_1(k_0.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2-\frac{1}{4}k_{\bar{1}}.k_{\bar{1}}(k_0.Y_{\bar{1}})^2(k_1.Y_1)^2-k_1.k_{\bar{1}}(k_0.Y_1)(k_0.Y_{\bar{1}})(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}}) \tag{7.2.39}
$$

It is gauge invariant under

$$
k_{1\mu}\to k_{1\mu}+\lambda_1 k_{0\mu};\qquad k_{\bar{1}\mu}\to k_{\bar{1}\mu}+\lambda_{\bar{1}}k_{0\mu}
$$

if we use the tracelessness condition on the gauge parameters:

$$
\lambda_1 k_1.k_{\bar{1}}k_{\bar{1}\mu}=\lambda_1 k_{\bar{1}}.k_{\bar{1}}k_{1\mu}=0=\lambda_{\bar{1}}k_1.k_{\bar{1}}k_{1\mu}=\lambda_{\bar{1}}k_1.k_1k_{\bar{1}\mu}
$$

Using (7.2.34) the EOM becomes:

$$
-\partial^2 S_{\mu\nu\rho\sigma}+\partial^\lambda\partial_{(\mu}S_{\nu)\lambda\rho\sigma}+\partial^\lambda\partial_{(\sigma}S_{|\mu\nu\lambda|\rho)}
$$

$$
-\partial_\mu\partial_\nu S^\lambda_{\lambda\rho\sigma}-\partial_\rho\partial_\sigma S_{\mu\nu}{}^\lambda{}_\lambda-\partial_{(\sigma}\partial_{(\nu}S_{\mu)}{}^\lambda{}_{\lambda|\rho)}=0 \tag{7.2.40}
$$

4. **Free Action**

It turns out that an action can also be written for this free theory:

$$
\begin{aligned}
S_{free}=-\frac{1}{2}S^{abcd}\Box S_{abcd}-\partial_a S^{aefg}\partial^b S_{befg}-\partial_a S^{efga}\partial^b S_{efgb}\\
-\partial_a\partial_b S^{abfg}S^c{}_{cfg}-\partial_a\partial_b S^{fgab}S_{fg}{}^c{}_c-4\partial_a\partial_b S^{eafb}S_e{}^c{}_{fc}\\
+\frac{1}{2}\left(S^c{}_c{}^{fg}\Box S^a{}_{afg}+S^{fgc}{}_c\Box S_{fg}{}^a{}_a+4S^{cf}{}_c{}^g\Box S^a{}_{fag}\right)\\
+2\left(S^c{}_c{}^{de}\partial_d\partial^b S_b{}^a{}_{ae}+S^{dec}{}_c\partial_e\partial^a S_d{}^b{}_{ba}\right)\\
-\frac{1}{2}\left(S^c{}_c{}^{de}\partial_e\partial^a S^b{}_{bad}+S^{dec}{}_c\partial_e\partial^a S_{ad}{}^b{}_b\right)
\end{aligned}
\tag{7.2.41}
$$

The EOM obtained from this action are linear combinations of (7.2.40) and its traces.

The action is of the form $S_aM^{ab}S_b$, where M is a symmetric in its indices. Its gauge variation is therefore $S_aM^{ab}\delta S_b$. Since $M^{ab}S_b$ is the EOM, which we know is gauge invariant, it must be true that $\delta(M^{ab}S_b)=M^{ab}\delta S_b=0$. Thus it follows that the action is also gauge invariant.

7.3 Interactions

We now turn to the issue of closed string interactions. The interactions are given by the second term in (5.3.21). It involves gauge invariant expression that we called a field strength because for Maxwell theory it is indeed the field strength.

7.3.1 Gauge Invariant Field Strength

We start with level 2. The interaction Lagrangian L at level 2 is best obtained by starting with the generalized loop variable, which we denote by $\mathcal{L}$.

$$\mathcal{L} = e^{i\left(k_0.Y+K_{1;0}Y_{1;0}+K_{0;\bar{1}}Y_{0;\bar{1}}+K_{1;\bar{1}}Y_{1;\bar{1}}+K_{2;0}Y_{2;0}+K_{0;\bar{2}}Y_{0;\bar{2}}+K_{1,1;0}Y_{1,1;0}+K_{0;\bar{1},\bar{1}}Y_{0;\bar{1},\bar{1}}+\dots\right)} \tag{7.3.42}$$

The field strength is given by:

$$\begin{aligned}
\frac{\delta}{\delta Y^\mu(z')}\int du\ \mathcal{L}(u) &= \int du\ \Big\{\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu(u)}\delta(u-z') + \frac{\partial\mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^\mu_1(u)}\partial_{x_1}\delta(u-z') \\
&+\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{\bar{1}}(u)}\partial_{\bar{x}_1}\delta(u-z') + \frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1,\bar{1}}(u)}\partial_{x_1}\partial_{\bar{x}_1}\delta(u-z')\Big\} \\
&= \int du\ \Big\{\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu(u)}\delta(u-z') - [\partial_{x_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_1(u)}]\delta(u-z') \\
&-[\partial_{\bar{x}_1}\frac{\partial\mathcal{L}[Y(u),Y_{n;\bar{m}}(u)]}{\partial Y^\mu_{\bar{1}}(u)}]\delta(u-z') + [\partial_{x_1}\partial_{\bar{x}_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1,\bar{1}}(u)}]\delta(u-z')+ \\
&[\partial^2_{x_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1,1;0}(u)}]\delta(u-z') + [\partial^2_{\bar{x}_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{0;\bar{1},\bar{1}}(u)}]\delta(u-z') \\
&-[\partial^2_{x_1}\partial_{\bar{x}_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1,1;\bar{1}}(u)}]\delta(u-z') - [\partial^2_{\bar{x}_1}\partial_{x_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1;\bar{1},\bar{1}}(u)}]\delta(u-z')\Big\} \\
&+[\partial^2_{\bar{x}_1}\partial^2_{x_1}\frac{\partial\mathcal{L}[Y(u),Y_{[n],[\bar{m}]}(u)]}{\partial Y^\mu_{1,1;\bar{1},\bar{1}}(u)}]\delta(u-z')
\end{aligned} \tag{7.3.43}$$

$$\begin{aligned}
&= \Big\{ik_{0\mu}\mathcal{L}(z') - iK_{1;0\mu}\partial_{x_1'}\mathcal{L}(z') - iK_{0;\bar{1}\mu}\partial_{\bar{x}_1'}\mathcal{L} + iK_{1;\bar{1}\mu}\partial_{x_1'}\partial_{\bar{x}_1'}\mathcal{L}(z')+ \\
&iK_{1,1;0\mu}\partial^2_{x_1}\mathcal{L} + iK_{0;\bar{1},\bar{1}}\partial^2_{\bar{x}_1}\mathcal{L} - iK^\mu_{1,1;\bar{1}}\partial^2_{x_1}\partial_{\bar{x}_1}\mathcal{L} - iK_{1;\bar{1},\bar{1}}\partial_{x_1}\partial^2_{\bar{x}_1}\mathcal{L} + iK_{1,1;\bar{1},\bar{1}\mu}\partial^2_{x_1}\partial^2_{\bar{x}_1}\mathcal{L}\Big\}
\end{aligned} \tag{7.3.44}$$

We have kept only terms that contribute to level $(1;\bar{1})$ and $(1,1;\bar{1},\bar{1})$. From the structure of $\mathcal{L}$ we can see that

$$\delta\mathcal{L} = \sum_{n,\bar{n}=1,2,\dots}(\lambda_n\frac{\partial}{\partial x_n}\mathcal{L} + \lambda_{\bar{n}}\frac{\partial}{\partial\bar{x}_n}\mathcal{L}) \tag{7.3.45}$$

Using (7.3.45) we can easily check that (7.3.44) is invariant under $\lambda_1,\lambda_{\bar{1}}$ variations, and at level 2, is the gauge invariant field strength for closed strings. We write it explicitly below:

$$\begin{aligned}
&-ik_{0\mu}(K_{1;0}.Y_{1;0})(K_{0;\bar{1}}.Y_{0;\bar{1}})e^{ik_0Y} - k_{0\mu}K_{1;\bar{1}}.Y_{1;\bar{1}}e^{ik_0Y} \\
&iK_{1;0\mu}(k_0.Y_{1;0})(K_{0;\bar{1}}.Y_{0;\bar{1}})e^{ik_0Y} + K_{1;0\mu}K_{0;\bar{1}}Y_{1,\bar{1}}e^{ik_0Y} \\
&iK_{0;\bar{1}\mu}(K_{1;0}.Y_{1;0})(k_0.Y_{0;\bar{1}})e^{ik_0Y} + K_{0;\bar{1}\mu}K_{1;0}Y_{1;\bar{1}}e^{ik_0Y}
\end{aligned}$$

$$-iK_{1;\bar{1}\mu}(k_0 Y_{0;\bar{1}})(k_0.Y_{1;0})e^{ik_0 Y} - K_{1;\bar{1}\mu}k_0.Y_{1;\bar{1}}e^{ik_0 Y} \tag{7.3.46}$$

The coefficient of $Y_1^\mu Y_{\bar{1}}^\nu$ can be seen to be

$$-k_{0\rho}k_{1\mu}k_{\bar{1}\nu} + k_{1\rho}k_{0\mu}k_{\bar{1}\nu} + k_{\bar{1}\rho}k_{1\mu}k_{0\nu} - K_{1;\bar{1}\rho}k_{0\mu}k_{0\nu} \tag{7.3.47}$$

In terms of space time fields this is

$$G_{\rho\mu\nu} \equiv \Big(-\partial_\rho(h_{\mu\nu} + B_{\mu\nu}) + \partial_\mu(h_{\rho\nu} + B_{\rho\nu}) + \partial_\nu(h_{\mu\rho} + B_{\mu\rho})\Big) - \partial_\mu\partial_\nu S_\rho$$

$$= \Gamma_{\rho\mu\nu} + H_{\rho\mu\nu} - \partial_\mu\partial_\nu S_\rho \tag{7.3.48}$$

7.3.2 Problems with massless spin 2 field strength

We have defined a field $S_\mu = \langle K_{1;\bar{1}\mu}\rangle$. However, at level 2 the physical fields are the graviton, antisymmetric tensor and dilaton. In fact since $K_{1;\bar{1}}$ involves $\bar{q}_1$, this field strength is well defined only if the graviton and dilaton are massive and $q_0 \neq 0$. The gauge transformation of S_μ is

$$\delta S_\mu = 2\epsilon_\mu$$

This means that it is a Stuckelberg field and one can fix a gauge such that it is zero. This would mean that the graviton has extra degrees of freedom - corresponding to a massive spin 2 field. While this is internally consistent this cannot describe the usual closed string states which are massless at this level. In the free equation of motion, only the combination $k_{0\mu}K^\mu_{1;\bar{1}}$ is involved. This was replaced by $k_1.k_{\bar{1}}$ and that solved the problem. For the interacting case we see that some drastic modification is called for.

Actually there is another problem. In the case of the open string we have seen that the theory continues to look Abelian even at the interacting level. The gauge transformation law is not modified by interactions. We have also pointed out that this may benot be unreasonable because we know that the gauge symmetry of the massless field is an Abelian U(1). For closed strings we know (from hindsight) that the massless spin 2 describes a graviton, which is “non Abelian” in the sense that the gauge transformation is different for the interacting theory.

If we combine this observation with the earlier one, a logical possibility is that both problems are solved if we modify our symmetry transformation rule so that we do not need a new field S_μ. This was done in [33].

7.3.3 Solution: Introduction of “Reference ”(or “Background”) metric and modification of symmetry transformation

The idea is to do two things: 1) Let us denote the loop variable gauge transformation for the massless fields by δ_G. Let us consider another transformation $\delta_T X^\mu = -\xi^\mu$. We attempt to make the action invariant under T. If $h_{\mu\nu}$ transforms as a 2 index symmetric tensor under this transformation then $h_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu$ is invariant.

$$\delta_T h_{\mu\nu} \equiv \xi^\lambda h_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu}h_{\lambda\nu} + \xi^\lambda{}_{,\nu}h_{\mu\lambda}; \qquad \delta_T X^\mu = -\xi^\mu$$

But T is not a symmetry of the theory because the *kinetic* term $\eta_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu$ is not invariant under this transformation.

$$\delta_T(\eta_{\mu\nu}\partial_z X^\mu \partial_{\bar{z}} X^\nu) = -\bar{\xi}_{(\mu,\nu)}\partial_z X^\mu \partial_{\bar{z}} X^\nu \tag{7.3.49}$$

Here $\bar{\xi}_\mu = \eta_{\mu\nu}\xi^\nu$. To remedy this we modify the kinetic term to

$$(\eta_{\mu\nu} + h^R_{\mu\nu})\partial_z X^\mu \partial_{\bar{z}} X^\nu \tag{7.3.50}$$

and assign to $h^R_{\mu\nu}$ the transformation law [6]

$$\delta_T h^R_{\mu\nu}(X) = \xi^\lambda h^R_{\mu\nu,\lambda} + \xi^\lambda{}_{,\mu}h^R_{\lambda\nu} + \xi^\lambda{}_{,\nu}h^R_{\mu\lambda} + \bar{\xi}_{(\mu,\nu)}; \qquad \delta_T X^\mu = -\xi^\mu(X) \tag{7.3.51}$$

[6] If we define a fully covariant $\xi_\mu = g^R_{\mu\nu}\xi^\nu$ then the same transformation law can be written as $\delta_T h^R_{\mu\nu}(X) = \nabla^R_{(\mu}\xi_{\nu)}$.

Now the kinetic term is also invariant. However we have made the action depend on an arbitrary quantity $h^R_{\mu\nu}$. So let us add the negative of this term to the interaction term, which becomes

$$(h_{\mu\nu} - h^R_{\mu\nu})\partial_z X^\mu \partial_{\bar{z}} X^\nu \tag{7.3.52}$$

While this term is not invariant under T, it is invariant under the combined action of G and T- provided we identify $\epsilon_\mu = \bar{\xi}_\mu$!The kinetic term is also invariant under $T + G$ because G does nothing to it and it was designed to be invariant under T. The transformation $T + G$ can be called a background general coordinate transformation. Under this both $h_{\mu\nu}$ and $h^R_{\mu\nu}$ transform as if they were metric fluctuations about $\eta_{\mu\nu}$ with the result that the difference $h - h^R$ transforms as an ordinary tensor.

What we have achieved is the following: By introducing a reference metric and including it in the action we have made the action invariant under a *new* symmetry. This is very close to ordinary general coordinate transformation (GCT) but it is not the same because it also transforms the reference metric. This seems pointless, because we want invariance under GCT which should act only on physical fields, and not on auxiliary constructs. However what saves the situation is that the full action does not depend on $h^R_{\mu\nu}$ because we have added and subtracted it out. Since our equations treat the kinetic term and interaction term separately, equations *will* depend on $h^R_{\mu\nu}$ at all intermediate stages of the calculation. However the final solution obtained after solving all the equations, should not depend on $h^R_{\mu\nu}$ because the starting point did not! Thus invariance under $G + T$ is equivalent to invariance under GCT - because the final answer depends only on the physical field $h_{\mu\nu}$ and on this they both have the same action.

7.3.4 Higher spin and Massive modes

In the above discussion we did not mention the massive modes. On the one hand they all have to be made covariant under $G + T$. This means introducing background covariant derivatives depending on $h^R_{\mu\nu}$. On the other hand we want the action to not depend on $h^R_{\mu\nu}$. So one must subtract some terms. Note that if $h^R_{\mu\nu}$ were equal to $h_{\mu\nu}$ the dependence would be allowed. So what must be subtracted out is a function of $h_{\mu\nu} - h^R_{\mu\nu}$. Thus for instance:

$$K_{n;\bar{m}\mu}\frac{D^2 Y^\mu}{Dx_n D\bar{x}_m} = K_{n;\bar{m}\mu}(\frac{\partial^2 Y^\mu}{\partial x_n \partial \bar{x}_m} + \Gamma^{R\mu}_{\rho\sigma} Y^\rho_n Y^\sigma_{\bar{m}})$$

We must subtract out the h^R dependent term - but not all of it. Thus

$$K_{n;\bar{m}\mu}(\frac{\partial^2 Y^\mu}{\partial x_n \partial \bar{x}_m} + \Gamma^{\mu}_{\rho\sigma} Y^\rho_n Y^\sigma_{\bar{m}})$$

is the required covariantization to be done when $h^R_{\mu\nu} = h_{\mu\nu}$. So we subtract [7]

$$K_{n;\bar{m}\mu}(\Gamma^{R\mu}_{\rho\sigma} - \Gamma^{\mu}_{\rho\sigma}) Y^\rho_n Y^\sigma_{\bar{m}} \tag{7.3.53}$$

from a higher massive mode vertex operator (which amounts to a field redefinition of the corresponding space time field):

$$k_{n\rho}k_{\bar{m}\sigma} Y^\rho_n Y^\sigma_{\bar{m}} \to [k_{n\rho}k_{\bar{m}\sigma} - K_{n;\bar{m}\mu}(\Gamma^{R\mu}_{\rho\sigma} - \Gamma^{\mu}_{\rho\sigma})] Y^\rho_n Y^\sigma_{\bar{m}}$$

Note also that (7.3.53) is a tensor under $G + T$. Thus the tensorial property of the redefined field $\langle k_{n\rho}k_{\bar{m}\sigma}\rangle$ is not affected.

These subtractions and field redefinitions of course have to be done at all levels in a systematic way. The result is a theory that is background covariant, but for which the full action does not depend on the reference metric - even though the kinetic and interaction term separately do.

[7] This differs from [33, 34] where the subtracted term was just $\Gamma^{R\mu}_{\rho\sigma} Y^\rho_n Y^\sigma_{\bar{m}}$ and not the difference. The present prescription is superior because the subtracted term is a tensor.

7.3.5 Choosing the reference metric equal to the physical metric

With this justification in mind, one can further simplify things by setting $h^R_{\mu\nu} = h_{\mu\nu}$. There are no subtractions to be made. Then the action depends only on $h^R_{\mu\nu}(= h_{\mu\nu})$ at all stages of the calculation, and the background symmetry $G+T$ which is now identified with GCT, is manifest at all stages of the calculation.

In non Abelian gauge theories background field methods have similar benefits. While gauge fixing makes the usual gauge symmetry non manifest, a new symmetry involving the background field is manifest. This restricts the form of the effective action. The background, which is arbitrary can then be chosen to be equal to the physical field. So the background gauge symmetry is the physical symmetry. So at all stages of the calculation the gauge symmtery is manifest. The utility of this method is described in an Appendix of [33] for the classical case, following the original discussion in [47] for the more complicated quantum case.

7.3.6 On $K_{1;\bar{1}\mu}$

Because $q_0 = 0$ the expression for $K_{1;\bar{1}\mu}$ does not make sense for the lowest level. So we will not use that expression at all. The role of $K_{1;\bar{1}\mu}$ is now played somehow by $h^R_{\mu\nu}$. We can try to understand this as follows. An integration by parts can be done can be done for the following term in the interaction Lagrangian:

$$\int d^2z\ K_{1;\bar{1}\mu}\partial_z\partial_{\bar{z}}Y^\mu e^{ik_0Y} = -\frac{1}{2}i(k_{0\nu}K_{1;\bar{1}\mu} + k_{0\mu}K_{1;\bar{1}\nu})\partial_z Y^\mu\partial_{\bar{z}}Y^\nu \tag{7.3.54}$$

Consider the gauge transformation $\delta K_{1;\bar{1}\mu} = 2\epsilon_\mu$. This gives for the interaction term a change:

$$-i(k_{0\nu}\epsilon_\mu + k_{0\mu}\epsilon_\nu)\partial_z Y^\mu\partial_{\bar{z}}Y^\nu e^{ik_0Y} \approx -\partial_{(\mu}\epsilon_{\nu)}\partial_z Y^\mu\partial_{\bar{z}}Y^\nu \tag{7.3.55}$$

But after the identification $\epsilon_\mu = \eta_{\mu\nu}\xi^\nu$, this is exactly the linearized transformation law for $-\int d^2z\, h^R_{\mu\nu}\partial_z Y^\mu\partial_{\bar{z}}Y^\nu$, the term that we added to the interaction Lagrangian. Thus the same role is played by a different term. The new field strength is gauge invariant : $\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}$. Thus we can set

$$\frac{1}{2}\langle k_{1(\mu}k_{\bar{1}\nu)} - (k_{0\nu}K_{1;\bar{1}\mu} + k_{0\mu}K_{1;\bar{1}\nu})\rangle = h_{\mu\nu} - h^R_{\mu\nu} \equiv \tilde{h}_{\mu\nu} \tag{7.3.56}$$

Let us use $\circ$ for the D+1 dimensional dot product and . for the D dimensional one. This combination is gauge invariant (under background transformations, where both h and h^R transform). However μ runs from 0 to D-1 in the above identification because the metric fluctuation $h_{\mu\nu}$ is not there in the D'th direction. The K-constraint is

$$k_0 \circ K_{1;\bar{1}} = k_1 \circ k_{\bar{1}} = k_1.k_1 + q_1\bar{q_1}$$

$$\implies k_0.K_{1;\bar{1}} + q_0 Q_{1;\bar{1}} - k_1.k_{\bar{1}} = q_1\bar{q_1}$$

$\langle q_1\bar{q_1}\rangle = \Phi_D$ is the dilaton. Also using (7.3.56) we get

$$-\tilde{h}^\mu_\mu + q_0 Q_{1;\bar{1}} = \Phi_D\ ; \mu = 0, .., D-1$$

Although q_0 is zero we have not set $q_0 Q_{1;\bar{1}}$ to zero. In fact if we use the usual expression for $K_{1;1\mu}$ we obtain that $q_0 Q_{1;\bar{1}} = q_1\bar{q_1}$. However as mentioned above, we do not wish to use that expression for the lowest level. So we leave it arbitrary. In fact we can use the K constraint to define $\langle q_0 Q_{1;\bar{1}}\rangle$ as $\tilde{h}^\mu_\mu + \Phi_D$. This equation relates the trace of the metric fluctuation to the dilaton in a gauge invariant way. We remind the reader that in the gauge fixed old covariant formulation, the trace of the graviton field is the dilaton.

From equation (7.3.56) we can write

$$\frac{1}{2}k_{1(\mu}k_{\bar{1}\nu)} = \tilde{h}^\mu_\mu + \frac{1}{2}k_{0(\mu}K_{1;\bar{1}\nu)}$$

Let us substitute this into the graviton free equation (B.2.12):

$$[-k_0 \circ k_0 k_{1\mu}k_{\bar{1}\nu} + k_0 \circ k_1 k_{0\mu}k_{\bar{1}\nu} + k_0 \circ k_{\bar{1}}k_{0\nu}k_{1\mu} - k_1 \circ k_{\bar{1}}k_{0\mu}k_{0\nu}] = 0 \tag{7.3.57}$$

In this equation all dot products range from 0 to D. Since $q_0 = 0$, in the first three terms we can drop q_0. The last term has a trace that includes $q_1\bar{q}_1$. We get ($\mu = 0, ..., D-1$.)

$$-k_0^2\tilde{h}_{\mu\nu} + k_{0\mu}k_0^\rho\tilde{h}_{\rho\nu} + k_{0\nu}k_0^\rho\tilde{h}_{\rho\mu} - k_{0\mu}k_{0\nu}(k_1 \circ k_{\bar{1}} - k_0.K_{1;\bar{1}}) = 0 \tag{7.3.58}$$

From the K-constraint $k_1 \circ k_{\bar{1}} - k_0.K_{1;\bar{1}} = q_0 Q_{1;\bar{1}} = \tilde{h}^\mu_\mu + \Phi_D$. Thus we get (we set $\Phi_D = 0$ for convenience):

$$-k_0^2\tilde{h}_{\mu\nu} + k_{0\mu}k_0^\rho\tilde{h}_{\rho\nu} + k_{0\nu}k_0^\rho\tilde{h}_{\rho\mu} - k_{0\mu}k_{0\nu}\tilde{h}^\rho{}_\rho = 0 \tag{7.3.59}$$

Let us write this equation as

$$k_0^\rho[-k_{0\rho}\tilde{h}_{\mu\nu} + k_{0\mu}\tilde{h}_{\rho\nu} + k_{0\nu}\tilde{h}_{\rho\mu}] - k_{0\mu}k_{0\nu}\tilde{h}^\rho{}_\rho = 0 \tag{7.3.60}$$

Viewed as a linearized equation for a perturbation about flat space this equation can also be written as

$$\partial^\rho(\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}) - \partial_\mu\partial_\nu\tilde{h}^\rho{}_\rho$$

If we interpret this as an equation in the RNC, we can immediately covariantize it to [8]

$$\nabla^{R\rho}[-\nabla^R_\rho\tilde{h}_{\mu\nu} + \nabla^R_\mu\tilde{h}_{\rho\nu} + \nabla^R_\nu\tilde{h}_{\rho\mu}] - \nabla^R_\mu\nabla^R_\nu\tilde{h}^\rho{}_\rho = 0 \tag{7.3.61}$$

This is to be compared with the linearized equation for the graviton fluctuation about a given background. We will check this in Section 7.6.3.

In addition to the gauge transformation, there is also the tensor rotation of $h^R_{\mu\nu}$ and $h_{\mu\nu}$. This is manifestly a symmetry of the action when the indices are contracted.

7.4 Covariant Description Summarized

In the above discussion we started with flat space and explained the problem of gauge invariance and the role of $h^R_{\mu\nu}$ in solving this problem. We then showed that the symmetry $G+T$ also called background covariance, is present in the action. We now summarize this by explaining the covariance starting directly from the curved space viewpoint. Thus we show that the kinetic term and interaction term are separately invariant under $G+T$. We then discuss how the ERG can be made manifestly invariant. The main tool is the Taylor expansion that can be done in an RNC and generalized, for scalar objects, to other coordinate systems.

7.4.1 Kinetic term

$$\int dz\ (\eta_{\mu\nu} + h^R_{\mu\nu}(Y))Y_1^\mu Y_{\bar{1}}^\nu = \int dz\ g^R_{\mu\nu}(Y(z))Y_1^\mu(z)Y_{\bar{1}}^\nu(z) \tag{7.4.62}$$

This term is manifestly invariant under background general coordinate transformations where $g^R_{\mu\nu}(Y)$ transforms like a metric tensor.

To do quantum calculations we need to Taylor expand $g^R_{\mu\nu}(Y)$ about a fixed point, Y_0, which we take to be the origin of the RNC. We will refer to the RNC as $\bar{Y}^\mu$ and let $\bar{Y}^\mu = 0$ be the origin. We use (A.3.17) to expand $\bar{g}^R_{\mu\nu}$ to get [45] (as usual the bar indicates that we are in the RNC):

$$\bar{g}^R_{\alpha\beta}(\bar{Y}) = \bar{g}^R_{\alpha\beta}(0) - \frac{1}{3}\bar{R}^R_{\alpha\mu\beta\lambda}(0)\bar{Y}^\mu\bar{Y}^\lambda - \frac{1}{3!}\bar{R}^R_{\alpha\gamma\beta\lambda,\mu}(0)\bar{Y}^\lambda\bar{Y}^\mu\bar{Y}^\gamma$$

$$+\frac{1}{5!}\{-6\bar{R}^R_{\alpha\delta\beta\gamma,\lambda\mu}(0) + \frac{16}{3}\bar{R}^R_{\lambda\beta\mu}{}^\rho(0)\bar{R}^R_{\gamma\alpha\delta\rho}(0)\}\bar{Y}^\lambda\bar{Y}^\mu\bar{Y}^\gamma\bar{Y}^\delta + ... \tag{7.4.63}$$

In the above expansion $\bar{g}^R_{\alpha\beta}(0)$ can be taken to be $\eta_{\alpha\beta}$, but for the moment we leave it as it is. The first derivative vanishes at the origin in the RNC. Thus the kinetic term in the RNC becomes

$$\bar{g}^R_{\mu\nu}(\bar{Y})\bar{Y}_1^\mu(z)\bar{Y}_{\bar{1}}^\nu(z) \tag{7.4.64}$$

[8] Note that while $\Gamma^\rho{}_{\mu\nu} - \Gamma^{R\rho}{}_{\mu\nu}$ is a tensor under background coordinate transformations, $\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}$ is not.

where we substitute (7.4.63) for $\bar{g}^R_{\mu\nu}(\bar{Y})$. This term as it stands is manifestly a scalar at the point P of the manifold and is easily generalized to a general coordinate system as

$$g^R_{\mu\nu}(Y)Y_1^\mu(z)Y_{\bar{1}}^\nu(z) \tag{7.4.65}$$

Here $Y_1^\mu(z)$ is a vector at the point P (with coordinates Y^μ) of the manifold as is clear from

$$\frac{\partial Y'^\mu(z)}{\partial z^\alpha} = \frac{\partial Y'^\mu(z)}{\partial Y^\nu(z)}\frac{\partial Y^\nu(z)}{\partial z^\alpha}$$

Now for the quantum treatment it is necessary to expand the metric in a Taylor series (7.4.63) so that we have an ordinary kinetic term that can be inverted to define green function, plus additional terms that will be interpreted as interactions. Each term in the Taylor expansion is a tensor *at the origin* O. Thus only if we interpret $Y_1^\mu(z)$ as a vector at the origin will it become a sum of scalars. But this is true because (see Appendix A (A)) the covariant derivative of a vector V^i is defined as

$$D_\beta V^i(z) \equiv \frac{\partial V^i(z)}{\partial z^\beta} + \Gamma^i_{ab}(X(z))\frac{\partial X^a(z)}{\partial z^\beta}V^b(z) \tag{7.4.66}$$

In the RNC at the origin, this becomes

$$\bar{D}_\beta V^i(z) \equiv \partial_\beta V^i(z)$$

because $\bar{\Gamma}^i_{ab}(0) = 0$. Thus $\bar{Y}_1^\mu(z) = \frac{\partial \bar{Y}^\mu}{\partial x_1}$ can be thought of as a vector at the origin (note that $\bar{Y}^\mu$ is a vector at the origin). Thus in a general coordinate system we will write $y^\mu(O)$ instead of $\bar{Y}^\mu$ and $\bar{Y}_1^\mu(z)$ will be written as

$$D_1 y^\mu(O)$$

The first term in the expansion of (7.4.65) becomes

$$\bar{g}^R_{\alpha\beta}(0)\bar{Y}_1^\alpha\bar{Y}_{\bar{1}}^\beta = g^R_{\alpha\beta}(O)D_1y^\alpha(O)D_{\bar{1}}y^\beta(O)$$

which is manifestly a scalar at the origin O with coordinates x_0.

Similarly the second term becomes

$$-\frac{1}{3}\bar{R}^R_{\alpha\mu\beta\lambda}(0)\bar{Y}^\mu(0)\bar{Y}^\lambda(0)\bar{Y}_1^\alpha(0)\bar{Y}_{\bar{1}}^\beta(0) = -\frac{1}{3}R^R_{\alpha\mu\beta\lambda}(O)y^\mu(O)y^\lambda(O)D_1y^\alpha(O)D_{\bar{1}}y^\beta(O) \tag{7.4.67}$$

which is a quartic interaction in the quantum world sheet theory. This will be included as part of L in the ERG.

Thus we have shown that the kinetic term can be written in a manifestly background covariant form. Let us now turn to the interaction term involving the massless graviton.

7.4.2 Regularization and Higher Derivative Kinetic Term

The Wilsonian ERG was originally formulated in momentum space. Regularization is achieved by keeping only low momentum modes in the theory. In the present formulation, position space is being used. Furthermore one has to be careful about not violating general coordinate invariance.

In position space one obvious way to regulate the theory is to add higher (world sheet) derivative terms to the kinetic term. If one wants a finite cutoff in (world sheet) position space then one has to work with a non local action (eg a lattice). In continuum language this becomes equivalent to adding arbitrarily high derivative terms to the kinetic term. This superficially seems to be in conflict with GCT because $\partial_z^2 X^\mu$ is not a vector unlike $\partial_z X^\mu$. One solution to this is to write $(D_z^R)^2 X^\mu$ where D_z^R is the covariant derivative introduced in Appendix A (A). The superscript R denotes that it is background covariant. In that case one can add terms of the form

$$\Delta S_0 = \int d^2z \sum_n c_n a^{2n}(\eta_{\mu\nu} + h^R_{\mu\nu})(D_z^R)^n X^\mu (D_{\bar{z}}^R)^n X^\nu$$

to the kinetic term action. The coefficients c_n characterize the regulator. Powers of a^n have been added to make the term dimensionless. Note that this introduces an additional dependence on the arbitrary $h^R_{\mu\nu}$.

In the loop variable formalism there is another option available. One can write

$$\Delta S_0 = \int d^2z \sum_{n,\bar{n}} c_{n,\bar{n}} a^{n+\bar{n}} (\eta_{\mu\nu} + h^R_{\mu\nu}) Y^\mu_n Y^\nu_{\bar{n}}$$

Unlike $\frac{\partial^n Y^\mu}{\partial x_1^n}$, $Y^\mu_n \equiv \frac{\partial Y^\mu}{\partial x_n}$ is a vector and the above term is background GCT invariant.

Thus as the above preliminary analysis indicates, it is possible to regulate the theory while maintaining general coordinate invariance. In this review we assume this is possible. As a result there is a well defined regularized Green function, for which one can write a covariant Taylor expansion.

If $h^R_{\mu\nu} = h_{\mu\nu}$ then any dependence in the action on $h^R_{\mu\nu}$ is allowed. Otherwise we have to subtract terms in the interaction Lagrangian to cancel the unwanted dependence on h^R introduced in the kinetic term. This involves adding terms of the form $\tilde{h}_{\mu\nu} Y^\mu_n Y^\nu_{\bar{n}}$. Such terms are already present in the action so this merely results in field redefinitions of massive fields.

In [33] some speculations were made on the possible new space time symmetries of string theory suggested by the presence of these terms. Since the ideas have not been developed sufficiently, we do not describe them in this review.

7.4.3 Massless Interactions

The graviton term in the world sheet theory can be written in terms of $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - h^R_{\mu\nu}$ in the RNC as

$$\int dz\ \tilde{h}_{\mu\nu}(\bar{Y}(z)) \bar{Y}^\mu_1(z) \bar{Y}^\nu_{\bar{1}}(z) = \int dz \int dk_0\ \tilde{h}_{\mu\nu}(k_0) e^{ik_0 \cdot \bar{Y}(z)} \bar{Y}^\mu_1(z) \bar{Y}^\nu_{\bar{1}}(z) \tag{7.4.68}$$

$\tilde{h}_{\mu\nu}(Y)$ is an ordinary tensor under background general coordinate transformation and thus this term is manifestly invariant. As was done above, this can be expanded in a Taylor series using (A.3.17). This is equivalent to expanding the exponential in powers of k_0. Thus (7.4.68) becomes (commas are covariant derivatives):

$$[\bar{\tilde{h}}_{\mu\nu}(0) + \bar{\tilde{h}}_{\mu\nu,\rho}(0)\bar{Y}^\rho(O) + \frac{1}{2!}\{\bar{\tilde{h}}_{\mu\nu,\rho\sigma}(0) - \frac{1}{3}(\bar{R}^{R\beta}_{\rho\mu\sigma}\bar{\tilde{h}}_{\beta\nu}(0) + \bar{R}^{R\beta}_{\rho\nu\sigma}\bar{\tilde{h}}_{\mu\beta}(0))\}\bar{Y}^\rho(O)\bar{Y}^\sigma(O) + ...]\bar{Y}^\mu_1(O)\bar{Y}^\nu_{\bar{1}}(O)$$

$$= [\tilde{h}_{\mu\nu}(0) + \tilde{h}_{\mu\nu,\rho}(0)y^\rho(O) + \frac{1}{2!}\{\tilde{h}_{\mu\nu,\rho\sigma}(0) - \frac{1}{3}(R^{R\beta}_{\rho\mu\sigma}\tilde{h}_{\beta\nu}(0) + R^{R\beta}_{\rho\nu\sigma}\tilde{h}_{\mu\beta}(0))\}y^\rho(O)y^\sigma(O) + ...]D_1 y^\mu(O) D_{\bar{1}} y^\nu(O) \tag{7.4.69}$$

Thus we have shown that the kinetic term and massless graviton vertex operator term in the world sheet action can be written in a manifestly background covariant form (i.e. invariant under $G+T$). For the massive modes Section 7.3.4 explains how manifest invariance under $G+T$ is achieved.

This concludes our discussion of how the world sheet action can be made manifestly invariant, starting from an action written in RNC.

7.5 ERG in curved space time

We now turn to the ERG which is also initially written using RNC. We show that it can also be written in a manifestly background covariant form. If both the world sheet action, and the ERG acting on it are manifestly symmetric, the resulting equations will also have the symmetry. In an actual calculation it is much easier to work in the RNC system. Thus the strategy will be to work in the RNC, obtain the equations, and then covariantize using the techniques of Section 6.

7.5.1 Covariantizing ERG

We reproduce the ERG equation here. This is written in flat space time. We will replace X^μ by $\bar{Y}^\mu$ when we are in curved space time.

$$\int du\ \frac{\partial L}{\partial \tau}\psi = \int dzdz'\ \{-\frac{1}{2}\underbrace{\dot{G}(z,z')G^{-1}(z,z')}_{field\ independent} -$$

$$\frac{1}{2}\dot{G}^{\mu\nu}(z,z')[\frac{\delta^2}{\delta X^\mu(z)\delta X^\nu(z')}\int du\ L[X(u),X'(u)]] + \frac{\delta}{\delta X^\mu(z)}\int du\ L[X(u)]\frac{\delta}{\delta X^\nu(z')}\int du'\ L[X(u')]\}\psi = 0 \tag{7.5.70}$$

Here $G^{\mu\nu}(z,z') \equiv \langle X^\mu(z)X^\nu(z')\rangle$. In curved space time X^μ is not a vector and the Green function does not have nice transformation properties. More precisely, the combination $X^\mu(z)\frac{\delta}{\delta X^\mu(z)}$ is what occurs in the ERG and this is not an invariant object. $\bar{Y}^\mu\frac{\delta}{\delta \bar{Y}^\mu(z)}$ on the other hand is a well defined object. In Appendix A (A) equation (A.6.32) defines

$$y^\mu(P) = \frac{\partial Y^\mu}{\partial \bar{Y}^\nu}|_P \bar{Y}^\mu(\bar{Y}_P)$$

a geometric object, namely the tangent vector to the geodesic at P. Thus $y^\mu(P)\frac{\delta}{\delta Y^\mu(z)}|_P$ is an invariant quantity. Thus

$$\langle y^\mu(P)y^\nu(P')\rangle\frac{\delta}{\delta Y^\mu(z)}|_P\frac{\delta}{\delta Y^\nu(z')}|_{P'}$$

is a well defined scalar object. When $z = z'$ it is the first term in the ERG, acting on L.

$$\int dz \int dz'\ \langle y^\mu(z)y^\nu(z)\rangle\frac{\delta^2}{\delta Y^\mu(z)\delta Y^\nu(z)}\int du\ L[u]$$

And when $z \neq z'$ it gives the second term

$$\int dz \int dz'\ \langle y^\mu(z)y^\nu(z')\rangle\frac{\delta}{\delta Y^\mu(z)}\int du\ L[u]\frac{\delta}{\delta Y^\nu(z')}\int du'\ L[u']$$

7.5.2 Interaction Term and Covariant OPE

The second term involves vertex operators at different points. Typically one performs an OPE to rewrite it as a sum of operators. The coefficient of any particular operator is the interaction term of an equation of motion for a field dual to that operator. The coefficient of that operator in the first term of the ERG provides the free part of the equation of motion for that field. It is thus essential to perform the OPE in a covariant way.

The issue of the covariant OPE was discussed in Section 6 in the context of open strings. The same issues are present for closed strings. The only new ingredient is that one has to expand in z and $\bar{z}$. We repeat some of the points here for convenience. There are two ingredients in an OPE: a Taylor expansion and a contraction. Let us illustrate this with the simplest example. [9]

$$e^{ikX(z)}e^{ipX(0)} = e^{ikX(z)+ipX(0)} = e^{ik(X(0)+z\partial_z X(0)+\bar{z}\partial_{\bar{z}}X(0)+\frac{1}{2}z^2\partial_z^2 X(0)+z\bar{z}\partial_z\partial_{\bar{z}}X(0)+\frac{1}{2}\bar{z}^2\partial_{\bar{z}}^2X(0)+...)+ipX(0)} \tag{7.5.71}$$

In order to take care of self contractions we can introduce the normal ordered vertex operators by

$$e^{ikX(z)} = e^{-\frac{k^2}{2}ln\ a^2} : e^{ikX(z)} : \tag{7.5.72}$$

[9]Let us take $\frac{1}{2\alpha'}\int d^2x\ (\nabla X)^2 = \frac{1}{\alpha'}\int d^2z\ \partial_z X\partial_{\bar{z}}X$ as our action. Then $\langle X(z)X(w)\rangle = -\frac{\alpha'}{2\pi}ln|\frac{z-w}{R}|$. We set $\alpha' = 4\pi$ for convenience.

and

$$e^{ikX(z)+ipX(0)} = e^{\frac{1}{2}\langle (ik.X(z)+ipX(0))(ik.X(z)+ipX(0))\rangle} : e^{ikX(z)+ipX(0)} :$$

$$= e^{-\frac{(k^2+p^2)}{2} ln\ a^2 - k.p\ ln\ (|z|^2+a^2)} : e^{ikX(z)+ipX(0)} :$$

$$= e^{-\frac{(k^2+p^2)}{2} ln\ a^2 - k.p\ ln\ (|z|^2+a^2)} : e^{ik(X(0)+z\partial_z X(0)+\bar{z}\partial_{\bar{z}} X(0)+\frac{1}{2}z^2\partial_z^2 X(0)+z\bar{z}\partial_z\partial_{\bar{z}} X(0)+\frac{1}{2}\bar{z}^2\partial_{\bar{z}}^2 X(0)..)+ipX(0)} : \tag{7.5.73}$$

We have used a choice of cutoff Green function $G(z,0;a) = ln\ (|z|^2+a^2)$ for illustration. These equations are written in flat space. In curved space time one has to be more careful. It has been argued in an earlier section that it is possible to regularize the theory on the world sheet while maintaining space time background general coordinate invariance. However non local expressions cannot easily be written in covariant form and the simplest way to get covariant expressions is to perform covariant Taylor expansions. Thus to begin with the Green function needs to be Taylor expanded. Thus $\langle y^\mu(z) y^\nu(0)\rangle$ has to be expressed as a power series in $z, \bar{z}$ and then each term has to be covariantized. The results are given in Appendix A (A). Thus for instance we write in the RNC (below symmetrization does not have a normalization factor of $n!$ - so that is explicitly multiplied.)

$$\bar{Y}^i(z) = \bar{Y}^i(0) + z^\alpha \bar{Y}^i_\alpha(0) + \frac{z^\alpha z^\beta}{2!}\partial_\alpha \bar{Y}^i_\beta(0) + \frac{z^\alpha z^\beta z^\gamma}{3!}\partial_\alpha\partial_\beta \bar{Y}^i_\gamma(0) + \frac{z^\alpha z^\beta z^\gamma z^\delta}{4!}\partial_\alpha\partial_\beta\partial_\gamma \bar{Y}^i_\delta(0) + ...$$

$$= \bar{Y}^i(0) + z^\alpha \bar{Y}^i_\alpha(0) + \frac{z^\alpha z^\beta}{2!} D_\alpha \bar{Y}^i_\beta(0) + \frac{z^\alpha z^\beta z^\gamma}{3!} D_\alpha D_\beta \bar{Y}^i_\gamma(0)$$

$$+\frac{z^\alpha z^\beta z^\gamma z^\delta}{4!}[D_\alpha D_\beta D_\gamma \bar{Y}^i_\delta(0) + \frac{1}{48}(R^i{}_{dac}(0) + R^i{}_{cad}(0))\bar{Y}^d_{(\delta}(0)\bar{Y}^c_\gamma(0)\bar{D}_\beta Y^a_{\alpha)}(0)] + ... \tag{7.5.74}$$

which is a covariant expansion.

Thus the Green function is expanded as

$$G^{ij}(z,0;a) = G^{ij}(0,0;a) + z^\alpha(\partial_\alpha G^{ij}(z,0;a))|_{z=0} + \frac{z^\alpha z^\beta}{2!}(\partial_\alpha\partial_\beta G^{ij}(z,0;a))|_{z=0} + ... \tag{7.5.75}$$

Note that every term is finite because of the presence of a cutoff. Each term involves the metric tensor, Riemann tensor and (covariant) derivatives thereof, all evaluated at one point, which can be taken to be the origin of the RNC. Similar expansions have to be done for the terms in the world sheet action which are products of space time fields and vertex operators. These are given in Appendix A (A). The covariant looking expansions are done in RNC, but the full object (product of field and vertex operators occurring in the interaction term of the ERG) is a scalar and the expansion is therefore valid in any coordinate system. Thus as an example (B.1.9) in Appendix A (A) gives the following result, which can be used in an OPE:

$$\begin{aligned} S_i(X(z))X^i_\alpha(z) \quad = \quad & S_i(X(0))X^i_\alpha(0) + z^\beta[\frac{1}{2}S_i(X(0))D_{(\beta}X^i_{\alpha)}(0) + \nabla_j S_i(X(0))X^i_\alpha X^j_\beta(0)] \\ & \frac{z^\beta z^\alpha}{2!}[\frac{1}{2}\nabla_j S_i(X(0))D_{(\beta}X^i_{\alpha)} + (\frac{\nabla_{(i}\nabla_{j)}S_i(X(0))}{2} + \frac{1}{3}R^l{}_{jki}(X(0))S_l(X(0)))X^k_\beta X^j_\gamma X^i_\alpha(0) + \\ & + \frac{D_{(\alpha}D_\beta X^i_{\gamma)}(0)S_i(X(0))}{6} + (\nabla_j S_i(X(0)))D_{(\beta}X_{\alpha)})^i X^j_\gamma(0)] \\ & + ... \end{aligned} \tag{7.5.76}$$

Thus to conclude: We have all the necessary ingredients for a covariant equation. Both the ERG and the action have been written in manifestly covariant form. We also have a covariant OPE. The result of all this is therefore covariant. Having assured ourselves that the result is covariant, we are free to work in the RNC where the equations are simpler, and covariantize everything in the end. This is computationally far simpler. In Section 6 we gave an algorithm for covariantizing the loop variable equation. This can be used.

Two points that need to be noted:

1. In Section 6 we had to perform some field redefinitions at the intermediate stages of the calculation in order to preserve the gauge invariance of the equations in curved space time. When the world sheet action is written in covariant form this step has to be incorporated and the same field redefinitions have to be performed. We will not bother to do this because we never actually use the covariant form of the world sheet action. Instead, are going to be working in the RNC with loop variables and the covariantization is only done at the last stage when we map to space time fields.

2. The metric $g^R_{\mu\nu}$ is completely arbitrary. So it is consistent to set it equal to the physical metric. In that case the field $\tilde{h}_{\mu\nu}$ that represents the graviton is actually zero. All interaction involving the gravitational field involves only the curvature tensor. Furthermore the symmetry $G+T$ which is manifest is the usual general coordinate invariance. For many purposes this choice is simpler and more convenient.

While keeping in mind the option of setting $\tilde{h}_{\mu\nu}=0$, we now proceed to write the background covariant form of the equations involving the graviton $\tilde{h}_{\mu\nu}$.

7.6 Background Covariant Equation for $\tilde{h}_{\mu\nu}$

We first evaluate the field strength corresponding to $\tilde{h}_{\mu\nu}$ which involves acting with the functional derivative (7.3.43) once. We write $S_{int}=\int duL[u]$. We work in the RNC and covariantize at the end.

7.6.1 Field Strength

The functional derivative below gives the field strength:

$$\frac{\delta S_{int}}{\delta \bar{Y}^{\rho}(z)} - \frac{\partial}{\partial x_1}\frac{\delta S_{int}}{\delta \bar{Y}_1^{\rho}(z)} - \frac{\partial}{\partial \bar{x}_1}\frac{\delta S_{int}}{\delta \bar{Y}_{\bar{1}}^{\rho}(z)} =$$

$$\frac{1}{2}[\frac{\partial \bar{\tilde{h}}_{\mu\nu}}{\partial \bar{Y}^{\rho}}(\bar{Y}(z)) - \frac{\partial \bar{\tilde{h}}_{\rho\nu}}{\partial \bar{Y}^{\mu}}(\bar{Y}(z)) - \frac{\partial \bar{\tilde{h}}_{\mu\rho}}{\partial \bar{Y}^{\nu}}(\bar{Y}(z)) + \frac{2}{3}\bar{Y}^{\beta}(z)(\bar{R}^R_{\rho\nu\mu\beta}(0)+\bar{R}^R_{\rho\mu\nu\beta}(0))]\bar{Y}_1^{\mu}(z)\bar{Y}_{\bar{1}}^{\nu}(z)$$

$$+[\bar{\tilde{h}}_{\rho\mu}(\bar{Y}(z)) + \frac{1}{6}\bar{Y}^{\alpha}(z)\bar{Y}^{\beta}(z)(\bar{R}^R_{\rho\alpha\mu\beta}(0)+\bar{R}^R_{\mu\alpha\rho\beta}(0))]\bar{Y}^{\mu}_{1;\bar{1}}(z) \qquad (7.6.77)$$

where $\bar{\tilde{h}}_{\mu\nu}=\bar{h}_{\mu\nu}-\bar{h}^R_{\mu\nu}$ and all arguments of fields have been displayed to avoid confusion. The bars on the metric fluctuation and curvature tensor are just to remind us that we are working in the RNC. The free indices are contracted with vertex operators and the second field strength in the interaction term. It then becomes a scalar. Anticipating this, to go to a general coordinate system we just remove the bars. The field strength tensor is, to this order,

$$F_{\rho\mu\nu}(\bar{Y}) = \frac{1}{2}[\frac{\partial \bar{\tilde{h}}_{\mu\nu}}{\partial \bar{Y}^{\rho}}(\bar{Y}(z)) - \frac{\partial \bar{\tilde{h}}_{\rho\nu}}{\partial \bar{Y}^{\mu}}(\bar{Y}(z)) - \frac{\partial \bar{\tilde{h}}_{\mu\rho}}{\partial \bar{Y}^{\nu}}(\bar{Y}(z)) + \frac{2}{3}\bar{Y}^{\beta}(z)(\bar{R}^R_{\rho\nu\mu\beta}(0)+\bar{R}^R_{\rho\mu\nu\beta}(0)) + ...] \qquad (7.6.78)$$

The field $\tilde{h}$, and the curvature tensor are gauge covariant and thus so is the field strength.

The first term involving $\tilde{h}$ is at a general point $\bar{Y}$ and has to be Taylor expanded about the origin in powers of $\bar{Y}$. The term involving the curvature tensor is already at O($\bar{Y}$) and is non leading. (It contributes to leading order in the free equation derived below.)

We can write the leading term in a general coordinate system by the usual procedure of writing background covariant derivatives:

$$(\Gamma_{\rho\mu\nu}-\Gamma^R_{\rho\mu\nu}) \rightarrow \frac{1}{2}(\nabla^R_{\mu}\tilde{h}_{\rho\nu}+\nabla^R_{\nu}\tilde{h}_{\rho\mu}-\nabla^R_{\rho}\tilde{h}_{\mu\nu}) \equiv \tilde{\Gamma}^R_{\rho\mu\nu} \qquad (7.6.79)$$

Similarly a non leading term (in powers of $\bar{Y}$) is

$$\frac{2}{3}\bar{Y}^{\beta}(z)(\bar{R}^R_{\rho\nu\mu\beta}(0)+\bar{R}^R_{\rho\mu\nu\beta}(0))$$

which can be covariantized to

$$\frac{2}{3}y^{\beta}(z)(R^{R}_{\rho\nu\mu\beta}(0)+R^{R}_{\rho\mu\nu\beta}(0)) \tag{7.6.80}$$

where y^{μ} was defined in (A.6.32) of Appendix A (A) and used in Sec 7.5 while covariantizing the ERG. In addition to this there are terms of order $O(\bar{Y})$ coming from the Taylor expansion of $\tilde{\Gamma}_{\rho\mu\nu}$

$$\bar{Y}^{\sigma}\bar{\nabla}^{R}\bar{\tilde{\Gamma}}_{\rho\mu\nu}(0)\to y^{\sigma}\nabla^{R}\tilde{\Gamma}_{\rho\mu\nu}(0) \tag{7.6.81}$$

The full field strength is a sum of all these: (7.6.79), (7.6.80) and (7.6.81) and higher order terms.

If we choose $h_{\mu\nu}=h^{R}_{\mu\nu}$ then the only term in the field strength is (7.6.80) and higher derivatives of the curvature tensor.

$$\frac{2}{3}y^{\beta}(z)(R_{\rho\nu\mu\beta}(0)+R_{\rho\mu\nu\beta}(0))+... \tag{7.6.82}$$

7.6.2 Free Equation

The free graviton equation for the metric fluctuation about flat space was derived in Section 7.3 and is (with the dilaton field $\Phi_D=0$):

$$\partial^{\rho}(\Gamma_{\rho\mu\nu}-\Gamma^{R}_{\rho\mu\nu})-\partial_{\mu}\partial_{\nu}\tilde{h}^{\rho}{}_{\rho}=0 \tag{7.6.83}$$

(7.6.83) is the RNC version (at the origin where $\Gamma^{R}(0)=0$) of the covariant equations in a general coordinate system:

$$\nabla^{R}_{\sigma}(g^{R\sigma\rho}\tilde{\Gamma}_{\rho\mu\nu})-\frac{1}{2}\nabla^{R}_{\mu}\nabla^{R}_{\nu}\tilde{h}^{\rho}{}_{\rho}=0 \tag{7.6.84}$$

where

$$\tilde{\Gamma}_{\rho\mu\nu}=\frac{1}{2}[-\nabla^{R}_{\rho}\tilde{h}_{\mu\nu}+\nabla^{R}_{\mu}\tilde{h}_{\rho\nu}+\nabla^{R}_{\nu}\tilde{h}_{\rho\mu}]$$

We work out for completeness the contribution due to the rest of the terms involving the background curvature tensor. The equation in loop variable notation is

$$\frac{1}{2}[-k_0^2k_{1\mu}k_{\bar{1}\nu}+k_0.k_1(k_{0\mu}k_{\bar{1}\nu}+k_{0\nu}k_{\bar{1}\mu})-k_{0\mu}k_{0\nu}k_1.k_{\bar{1}}]=0 \tag{7.6.85}$$

The equation is being evaluated at the origin O, where $\bar{Y}^{\mu}=0$, so only the quadratic term contributes - the cubic and higher order terms do not contribute.

Therefore we use

$$k_{1\mu}k_{\bar{1}\nu}=\frac{1}{6}\bar{Y}^{\alpha}\bar{Y}^{\beta}(\bar{R}^{R}_{\mu\alpha\nu\beta}(0)+\bar{R}^{R}_{\nu\alpha\mu\beta}(0))$$

in the above and obtain

$$-\bar{R}^{R}_{\mu\nu}\bar{Y}_1^{\mu}\bar{Y}_{\bar{1}}^{\nu} \tag{7.6.86}$$

Thus the total for the graviton contribution to the free graviton EOM is (dropping bars):

$$(R^{R}_{\mu\nu}+\nabla^{R}_{\sigma}(g^{R\sigma\rho}\tilde{\Gamma}_{\rho\mu\nu})-\frac{1}{2}\nabla^{R}_{\mu}\nabla^{R}_{\nu}\tilde{h}^{\rho}{}_{\rho})Y_1^{\mu}Y_{\bar{1}}^{\nu} \tag{7.6.87}$$

7.6.3 Comparison with Einstein's Equation

This free equation in the first case should be compared with what one expects for a graviton from Einstein's vacuum equation $R_{\mu\nu}=0$ expanded to linear order in $\tilde{h}$, about a background. One can expand as follows:

$$R_{\mu\nu}=R^{R}_{\mu\nu}+\delta R_{\mu\nu} \tag{7.6.88}$$

To evaluate δR, go to an inertial frame with $\Gamma=0$ at the point under consideration,

$$R^{\alpha}{}_{\mu\beta\nu}=\partial_{\beta}\Gamma^{\alpha}{}_{\mu\nu}-\partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta}$$

So

$$\delta R^{\alpha}_{\ \mu\beta\nu} = \partial_\beta \delta\Gamma^{\alpha}_{\ \mu\nu} - \partial_\nu \delta\Gamma^{\alpha}_{\ \mu\beta}$$

Now unlike $\Gamma^{\alpha}_{\ \mu\nu}$, $\delta\Gamma^{\alpha}_{\ \mu\nu}$ is a tensor, so the above equation, if written covariantly, is valid in all frames:

$$\delta R^{\alpha}_{\ \mu\beta\nu} = \nabla_\beta \delta\Gamma^{\alpha}_{\ \mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\ \mu\beta} \tag{7.6.89}$$

So we get the Palatini equation:

$$\delta R_{\mu\nu} = \delta R^{\alpha}_{\ \mu\alpha\nu} = \nabla_\alpha \delta\Gamma^{\alpha}_{\ \mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\ \mu\alpha} \tag{7.6.90}$$

We now show that to linear order in $\tilde{h}$ (or h),

$$\delta\Gamma^{\alpha}_{\ \mu\nu} \equiv \Gamma^{\alpha}_{\ \mu\nu} - \Gamma^{R\alpha}_{\ \mu\nu} = g^{R\alpha\rho}\tilde{\Gamma}_{\rho\mu\nu} \tag{7.6.91}$$

where

$$\frac{1}{2}(\nabla^R_\mu \tilde{h}_{\rho\nu} + \nabla^R_\nu \tilde{h}_{\rho\mu} - \nabla^R_\rho \tilde{h}_{\mu\nu}) \equiv \tilde{\Gamma}^R_{\rho\mu\nu}$$

Writing $g^{\rho\sigma} = g^{R\rho\sigma} + \delta g^{\rho\sigma}$ we get

$$\Gamma^{\sigma}_{\ \mu\nu} - \Gamma^{R\sigma}_{\ \mu\nu} = g^{R\rho\sigma}(\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}) + \delta g^{\rho\sigma}\Gamma_{\rho\mu\nu}$$

$$= g^{R\rho\sigma}(\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}) + \delta g^{\rho\sigma}\Gamma^R_{\rho\mu\nu} \tag{7.6.92}$$

to linear order in h.

Now consider the RHS of (7.6.91). Expand the covariant derivatives:

$$g^{R\rho\sigma}[(\Gamma_{\rho\mu\nu} - \Gamma^R_{\rho\mu\nu}) - \Gamma^{R\alpha}_{\ \mu\nu}\tilde{h}_{\rho\alpha}] \tag{7.6.93}$$

If we now take into account the fact that $\delta g^{\rho\sigma} = -g^{R\rho\alpha}h_{\alpha\beta}g^{R\beta\sigma}$ we see that (7.6.92) and (7.6.93) are equal and we have the result (7.6.91). Furthermore taking the trace of (7.6.91) we get

$$\Gamma^{\alpha}_{\ \mu\alpha} - \Gamma^{R\alpha}_{\ \mu\alpha} = g^{R\alpha\rho}\frac{1}{2}(\nabla^R_\mu \tilde{h}_{\rho\alpha} + \nabla^R_\alpha \tilde{h}_{\mu\rho} - \nabla^R_\rho \tilde{h}_{\mu\alpha}) = \frac{1}{2}\nabla^R_\mu \tilde{h}^{\rho}_{\ \rho} \tag{7.6.94}$$

Inserting (7.6.91) and (7.6.94) into (7.6.90) we obtain the equation for $\tilde{h}$ in the background metric (including for completeness the background contribution):

$$R^R_{\mu\nu} + \nabla^R_\sigma(g^{R\sigma\rho}\tilde{\Gamma}_{\rho\mu\nu}) - \frac{1}{2}\nabla^R_\nu \nabla^R_\mu \tilde{h}^{\rho}_{\ \rho} = 0 \tag{7.6.95}$$

which is the covariantized equation that we obtained in the loop variable approach in Section 7.6.2, (7.6.87).

Finally if we set $h_{\mu\nu} = h^R_{\mu\nu}$ (so that $\tilde{h}_{\mu\nu} = 0$) then the free equation simply reduces to Einstein's equation

$$R_{\mu\nu} = 0$$

7.6.4 Interactions of the graviton

The interaction terms will involving OPE of the field strength with itself as well as with field strengths of other modes. The field strength was given to leading order in Section 7.6.1 ((7.6.79),(7.6.80),(7.6.81)). One has to expand the OPE covariantly. This has been described in Section 7.5.2 and can be applied here in a fairly obvious way. We will not describe this again.

In the last few subsections we have described at some length the contribution of the graviton to the equation of motion. There was some subtlety in this because of the dual role of the graviton: On the one hand it is just another mode of the closed string and on the other, being massless one has to describe in a convenient way the gauge symmetry associated with it - which is general covariance.

The massive modes are more straightforward from this point of view. Nevertheless one has to resolve the clash between general covariance of their EOM and the higher "broken" gauge symmetries associated with them. This clash, at the algebraic level, is the same as the one we faced for the open string modes for which a solution was presented in Section 6. The solution was given there as a four step algorithm and the same algorithm can be applied *mutatis mutandis* for closed string massive modes also.

7.7 Example: Closed String - Level $(2,\bar{2})$

In this section we work out the result for the first massive level of closed strings. The free equation of motion and free action were worked out in Section 7.2.2. We give here the free equation in a curved space time background using the technique for mapping from loop variable expressions to fields in curved space time described in Section 6. The loop variable equation of motion is

$$-\frac{1}{4}k_0^2(k_1.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_1(k_0.Y_1)(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_{\bar{1}}(k_0.Y_{\bar{1}})(k_{\bar{1}}.Y_{\bar{1}})(k_1.Y_1)^2+$$

$$-\frac{1}{4}k_1.k_1(k_0.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2-\frac{1}{4}k_{\bar{1}}.k_{\bar{1}}(k_0.Y_{\bar{1}})^2(k_1.Y_1)^2-k_1.k_{\bar{1}}(k_0.Y_1)(k_0.Y_{\bar{1}})(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}})=0 \qquad (7.7.96)$$

It is gauge invariant under

$$k_{1\mu}\to k_{1\mu}+\lambda_1 k_{0\mu};\qquad k_{\bar{1}\mu}\to k_{\bar{1}\mu}+\lambda_{\bar{1}}k_{0\mu}$$

if we use the tracelessness condition on the gauge parameters:

$$\lambda_1 k_1.k_{\bar{1}}k_{\bar{1}\mu}=\lambda_1 k_{\bar{1}}.k_{\bar{1}}k_{1\mu}=0=\lambda_{\bar{1}}k_1.k_{\bar{1}}k_{1\mu}=\lambda_{\bar{1}}k_1.k_1 k_{\bar{1}\mu} \qquad (7.7.97)$$

The fields were also defined in Sec 7.2. We specialize to the four index tensor and define the dimensionally reduced fields:

$$\begin{aligned}
\langle k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle &= S_{1\mu1\nu\bar{1}\rho\bar{1}\sigma}\\
\langle q_1 k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\rangle &= S_{11\nu\bar{1}\rho\bar{1}\sigma}q_0\\
\langle q_{\bar{1}}k_{1\mu}k_{1\nu}k_{\bar{1}\rho}\rangle &= S_{1\mu1\nu\bar{1}\rho1}q_0\\
&...
\end{aligned} \qquad (7.7.98)$$

$$(7.7.99)$$

and similarly for the remaining fields. We hope the notation is clear to the reader. Consider the first term in (7.7.96) (the bar on Y indicates RNC):

$$k_0^2(k_1.\bar{Y}_1)^2(k_{\bar{1}}.\bar{Y}_{\bar{1}})^2=k_0^2k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}\bar{Y}_1^{\mu}\bar{Y}_1^{\nu}\bar{Y}_{\bar{1}}^{\rho}\bar{Y}_{\bar{1}}^{\sigma} \qquad (7.7.100)$$

Thus we need to map $k_0^2k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}$ to a space-time field using our modified prescription.

The four index tensor equation map is quite tedious to work out. There is no new complication that arises except that we need the Taylor expansion in RNC (A.3.17) to higher orders. So we will only give outlines.

The constraints (A.2.8) can be mapped directly to space-time field constraints. If it is zero in flat space, it continues to be zero even in curved space since the extra curvature couplings in curved space are also linear in the constraint.

Step 1

Let $k_{1\mu}=\tilde{k}_{1\mu}+y_1k_{0\mu}$ and $k_{\bar{1}\mu}=\tilde{k}_{\bar{1}\mu}+y_{\bar{1}}k_{0\mu}$. Then we obtain:

$$k_0^2k_{1\mu}k_{1\nu}k_{\bar{1}\rho}k_{\bar{1}\sigma}=k_0^2(\tilde{k}_{1\mu}+y_1k_{0\mu})(\tilde{k}_{1\nu}+y_1k_{0\nu})(\tilde{k}_{\bar{1}\rho}+y_{\bar{1}}k_{0\rho})(\tilde{k}_{\bar{1}\sigma}+y_{\bar{1}}k_{0\sigma}) \qquad (7.7.101)$$

This has to be done for each term in (7.7.96).

Step 2

We define some tilde fields at the intermediate stage as:

$$\begin{aligned}
\langle \tilde{k}_{1\mu}\tilde{k}_{1\nu}\tilde{k}_{\bar{1}\rho}\tilde{k}_{\bar{1}\sigma}\rangle &= \tilde{S}_{1\mu1\nu\bar{1}\rho\bar{1}\sigma}\\
\langle y_1\tilde{k}_{1\nu}\tilde{k}_{\bar{1}\rho}\tilde{k}_{\bar{1}\sigma}\rangle &= \tilde{S}_{11\nu\bar{1}\rho\bar{1}\sigma}\\
\langle y_{\bar{1}}\tilde{k}_{1\mu}\tilde{k}_{1\nu}\tilde{k}_{\bar{1}\rho}\rangle &= \tilde{S}_{1\mu1\nu\bar{1}\bar{1}\rho}\\
\langle y_1y_{\bar{1}}\tilde{k}_{1\mu}\tilde{k}_{\bar{1}\sigma}\rangle &= \tilde{S}_{11\mu\bar{1}\bar{1}\sigma}\\
\langle y_{\bar{1}}^2\tilde{k}_{1\mu}\tilde{k}_{1\nu}\rangle &= \tilde{S}_{1\mu1\nu\bar{1}\bar{1}}\\
&...
\end{aligned} \qquad (7.7.102)$$

etc.

We need to work out the map between these sets of fields. We have

$$\langle q_1 q_1 \bar{q}_{\bar{1}} \bar{q}_{\bar{1}} \rangle = S_{11\bar{1}\bar{1}} = q_0^4 \langle y_1 y_1 y_{\bar{1}} y_{\bar{1}} \rangle = q_0^4 \tilde{S}_{11\bar{1}\bar{1}} \tag{7.7.103}$$

$$\begin{aligned} \langle q_1 q_1 \bar{q}_{\bar{1}} k_{\bar{1}\rho} \rangle &= q_0^3 \langle y_1 y_1 y_{\bar{1}} (\tilde{k}_{\bar{1}\rho} + y_{\bar{1}} k_{0\rho}) \rangle \\ \implies S_{11\bar{1}\bar{1}\rho} &= q_0^3 \tilde{S}_{11\bar{1}\bar{1}\rho} + q_0^3 \nabla_\rho \tilde{S}_{11\bar{1}\bar{1}} \end{aligned} \tag{7.7.104}$$

$$\begin{aligned} \langle q_1 q_1 k_{\bar{1}\rho} k_{\bar{1}\sigma} \rangle &= q_0^2 y_1^2 \langle \tilde{k}_{1\rho} \tilde{k}_{\bar{1}\sigma} + y_{\bar{1}} k_{0\rho} \tilde{k}_{\bar{1}\sigma} + y_{\bar{1}} k_{0\sigma} \tilde{k}_{\bar{1}\rho} + y_{\bar{1}}^2 k_{0\rho} k_{0\sigma} \rangle \\ S_{11\bar{1}\rho\bar{1}\sigma} &= q_0^2 \tilde{S}_{11\bar{1}\rho\bar{1}\sigma} + q_0^2 \nabla_\rho \tilde{S}_{11\bar{1}\bar{1}\sigma} + q_0^2 \nabla_\sigma S_{11\bar{1}\bar{1}\rho} + q_0^2 \nabla_\rho \nabla_\sigma \tilde{S}_{11\bar{1}\bar{1}} \end{aligned} \tag{7.7.105}$$

Solving these equation for the tilde fields one obtains:

$$\begin{aligned} \tilde{S}_{11\bar{1}\bar{1}} &= \frac{1}{q_0^4} S_{11\bar{1}\bar{1}} \\ \tilde{S}_{11\mu\bar{1}\bar{1}} &= \frac{S_{11\mu\bar{1}\bar{1}}}{q_0^3} - \frac{\nabla_\mu S_{11\bar{1}\bar{1}}}{q_0^4} \\ \tilde{S}_{11\bar{1}\rho\bar{1}\sigma} &= \frac{S_{11\bar{1}\rho\bar{1}\sigma}}{q_0^2} - \frac{\nabla_{(\rho} S_{11\bar{1}\bar{1}\sigma)}}{q_0^3} + \frac{\nabla_\rho \nabla_\sigma S_{11\bar{1}\bar{1}}}{q_0^4} \end{aligned} \tag{7.7.106}$$

The above equations are essentially the same as was given in the last section for open strings. We further need expressions for the three and four index tensors.

After some straightforward algebra one finds the following relation for the three index tensor:

$$\begin{aligned} \tilde{S}_{11\nu\bar{1}\rho\bar{1}\sigma} &= \frac{S_{11\nu\bar{1}\rho\bar{1}\sigma}}{q_0} - \frac{1}{q_0^2}[\nabla_\nu S_{11\bar{1}\rho\bar{1}\sigma} + \nabla_\rho S_{11\nu\bar{1}\bar{1}\sigma} + \nabla_\sigma S_{11\nu\bar{1}\rho\bar{1}}] \\ &+ \frac{1}{q_0^3}[\nabla_\rho \nabla_\nu S_{11\bar{1}\bar{1}\sigma} + \nabla_\sigma \nabla_\nu S_{11\bar{1}\bar{\rho}1} + \nabla_\sigma \nabla_\rho S_{11\nu\bar{1}\bar{1}}] + \frac{1}{q_0^4} \nabla_\sigma \nabla_\nu \nabla_\rho S_{11\bar{1}\bar{1}} \\ &+ \frac{2}{3}(R^\lambda{}_{\rho\nu\sigma} + R^\lambda{}_{\sigma\nu\rho})[\frac{S_{11\bar{1}\bar{1}\lambda}}{q_0^3} - \frac{\nabla_\lambda S_{11\bar{1}\bar{1}}}{q_0^4}] + \frac{1}{3}(R^\lambda{}_{\sigma\rho\nu} + R^\lambda{}_{\nu\rho\sigma})[\frac{S_{11\lambda\bar{1}\bar{1}}}{q_0^3}] \end{aligned} \tag{7.7.107}$$

Finally using the same methods the four index tensor is seen to satisfy a relation of the form

$$S_{1\mu 1\nu\bar{1}\rho\bar{1}\sigma} = \tilde{S}_{1\mu 1\nu\bar{1}\rho\bar{1}\sigma} + (\textit{lower index tensors})$$

Using (7.7.106) and (7.7.107), one can solve for $\tilde{S}_{1\mu 1\nu\bar{1}\rho\bar{1}\sigma}$ in terms of the ordinary fields. We do not work it out here. Thus we have expressions for the tilde fields in terms of original fields. This is the end of Step 2.

Step 3

Using the results of (A.3.17) we obtain for instance:

$$\begin{aligned} \langle k_0^2 \tilde{k}_{1\mu} \tilde{k}_{1\nu} \tilde{k}_{\bar{1}\rho} \tilde{k}_{\bar{1}\sigma} \rangle &= \nabla^2 \tilde{S}_{11\bar{1}\bar{1}\mu\nu\rho\sigma} - \frac{1}{3}(R^\lambda{}_\mu \tilde{S}_{11\bar{1}\bar{1}\lambda\nu\rho\sigma} + R^\lambda{}_\nu \tilde{S}_{11\bar{1}\bar{1}\mu\lambda\rho\sigma} + R^\lambda{}_\rho \tilde{S}_{11\bar{1}\bar{1}\mu\nu\lambda\sigma} + R^\lambda{}_\sigma \tilde{S}_{11\bar{1}\bar{1}\mu\nu\rho\lambda}) \\ \langle k_0^2 y_1 k_{0\mu} \tilde{k}_{1\nu} \tilde{k}_{\bar{1}\rho} \tilde{k}_{\bar{1}\sigma} \rangle &= \nabla_\mu \nabla^2 \tilde{S}_{11\bar{1}\bar{1}\nu\rho\sigma} - (R^\lambda{}_\nu \nabla_\mu \tilde{S}_{11\bar{1}\bar{1}\lambda\rho\sigma} + R^\lambda{}_\rho \nabla_\mu \tilde{S}_{11\bar{1}\bar{1}\nu\lambda\sigma} + R^\lambda{}_\sigma \nabla_\mu \tilde{S}_{11\bar{1}\bar{1}\nu\rho\lambda}) \\ &- \frac{1}{2}(\nabla_\mu R^\lambda{}_\nu \tilde{S}_{11\bar{1}\bar{1}\lambda\rho\sigma} + \nabla_\mu R^\lambda{}_\rho \tilde{S}_{11\bar{1}\bar{1}\nu\lambda\sigma} + \nabla_\mu R^\lambda{}_\sigma \tilde{S}_{11\bar{1}\bar{1}\nu\rho\lambda}) \end{aligned} \tag{7.7.108}$$

We do not bother to write down the rest of the terms. As the number of derivatives increase the expressions become more complicated. Hopefully it is clear to the reader that given the taylor expansion (A.3.17) the terms can easily be written down.

Step 4

The last step is to plug in the results of Step 2 into (7.7.108) and obtain expressions involving the original fields and then insert these into (7.7.101). Since the map from loop variables to space time fields has been done in such a way that gauge transformations are well defined, the gauge invariance of the loop variable expression guarantees the gauge invariance of the field theory expression.

This has to be done for each term in the equation of motion (7.7.96). We do not work out the details since the details are not very illuminating and there are no further conceptual issues.

It should also be clear that all higher derivative terms not involving the curvature tensor cancel and reproduce the flat space result. Then the curvature couplings to the four index tensor field give the naive covariantization just as in (6.3.34). The remaining terms are curvature coupling to Stuckelberg fields. These involve higher derivatives and are required for gauge invariance. However they can be set to zero by a choice of gauge and so propagation of physical fields is described by a second order differential equation. Presumably this is sufficient for classical consistency of the theory.

7.8 Central Charge

In the BRST formulation of string theory, which is a gauge invariant one, the extra dimension corresponds to a bosonized ghost field field on the world sheet. Accordingly the central charge contribution of this is -26. This is cancelled by a contribution of +26 from 26 space time coordinates. This cancellation is crucial for gauge invariance. Furthermore although formally an extra dimension, in detail the ghost has different dynamics. Not only that, the equations of motion for the massive fields are not in a form that would be obtained by dimensional reduction of a massless theory in one higher dimension. In [30] it was also shown that *in 26 dimensions* it is possible to do field redefinitions that bring the free equations of motion into standard form - i.e. of a dimensionally reduced massless theory.

In the loop variable approach the viewpoint is that space time gauge invariance is the primary requirement. The construction described in the earlier sections achieves this requirement. However the extra coordinate behaves more or less like an extra dimension. It plays the role of producing the effects of the naive world sheet scaling dimension in terms of an anomalous dimension q_0^2 where q_0 is the momentum in the extra dimension. But if this is not to affect the value of the poles of the scattering amplitudes it must be true that the correlation function of these field must vanish except at coincident points. Thus it must behave like a massive (on the world sheet) field. A priori it is not clear what the connection is with the bosonized ghost field. Furthermore it is not clear why the central charge should be -26 and hence there is no critical dimension. It seems natural to conclude that only in 26 dimensions is this theory equivalent to critical string theory. In fact in [40] it was shown (and reviewed here in Section 8) that at least for level 2 and 3 open string modes, it is possible to redefine fields such that the physical state constraints and gauge transformations in the loop variable approach have exactly the same form as in string theory old covariant formalism *provided the critical dimension is 26 and the masses of the fields have the string theory value.*

In other dimensions the theory can still be gauge invariant and have massless gauge fields, so it does not seem to have the properties usually ascribed to non critical strings. We do not have an answer to this question. Nevertheless there is a more limited sense in which the question can be asked. In the world sheet RG approach to string theory the critical dimension constraint arises as an equation of motion for the dilaton whose vertex operator is taken as $\partial_z\partial_{\bar z}\sigma$. The EOM for the dilaton thus has a term $D-26$ in it. Thus we can at least ask about the dilaton equation in the loop variable formalism and whether the central charge term arises. We give a partial answer to this question.

In Section 5.3 we saw that the ERG has a field independent term $\frac{1}{2}Tr\dot{G}G^{-1}$. This gives the contribution to the trace anomaly from the determinant (or equivalently the measure in Fujikawa's interpretation). It has non universal divergent parts and a universal finite part proportional to $\partial_z\partial_{\bar z}\sigma$ where σ is the Liouville mode and gives the central charge.

The connection to the dilaton comes from the observation [2, 48] that normal ordering operators such as $\partial_z X\partial_{\bar z}X$ produces $\partial_z\partial_{\bar z}\sigma$ on a curved world sheet. This is the form of the trace of the kinetic term and the coefficient of this is the dilaton. Alternatively in the loop variable approach one can take operators of the form $Q_{1;\bar 1}\partial_z\partial_{\bar z}\theta e^{iq_0\theta}$ which on normal ordering produces $q_0 Q_{1;\bar 1}\partial_z\partial_{\bar z}\sigma$. Thus we can expect $q_0 Q_{1;\bar 1}$ to stand

for the dilaton. In fact we have seen that it stands for $\tilde{h}^{\rho}_{\rho} + \Phi_D$. So if we set $h_{\mu\nu} = h^R_{\mu\nu}$ (so that $\tilde{h} = 0$), then indeed it stands for the dilaton operator. Thus we can expect the dilaton equation to include a contribution from the central charge. The precise connection between the bosonized ghost and the extra coordinate in the loop variable formalism needs to be clarified in order to find the central charge.

8 Connection with the Old Covariant Formalism

In this section we review some results that have been worked out in [40] on the relation between the loop variable method and the old covariant formalism that was alluded to in the last section [10] This can be viewed as the first steps towards a proof that in the critical dimension, the two approaches describe the same physical theory. The discussion is confined to the open string and that too for the first two massive levels. Hopefully it can be generalized.

8.1 Old Covariant Formalism

In the OC formalism the physical state constraints are given by the action of $L_{+n}, n \geq 0$ and gauge transformations by the action of $L_{-n}, n > 0$. In [41] a closed form expression is given for the following:

$$e^{\sum_n \lambda_{-n} L_{+n}} e^{i\sum_n k_n Y_n} |0\rangle \tag{8.1.109}$$

where $\tilde{Y}_n = \frac{\partial_z^n X(z)}{(n-1)!}$, $\tilde{Y}_0 = \tilde{Y} = X$. We will need it mainly to linear order in λ_n which can be obtained from:

$$e^{-\frac{1}{2}\mathcal{Y}^{\mathcal{T}}\lambda\mathcal{Y}} e^{i\sum_n k_n Y_n} \tag{8.1.110}$$

where $\mathcal{Y}^T = (.., \tilde{Y}_3, \tilde{Y}_2, \tilde{Y}_1, -ik_0, -ik_1, -2ik_2, -3ik_3, ...)$ and λ is a matrix whose elements are given by:

$$(\lambda)_{m,n} = \lambda_{m+n}$$

(8.1.110) will be used below.

8.1.1 Level 2

Vertex operators

The level two vertex operators are obtained from

$$e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2}|0\rangle = e^{ik_0.X}(... - \frac{1}{2}k_{1\mu}k_{1\nu}\partial X^{\mu}\partial X^{\nu} + ik_{2\mu}\partial^2 X^{\mu} + ...)|0\rangle \tag{8.1.111}$$

Action of $L_{\pm n}$

Using (8.1.110) we get

$$\begin{aligned} exp \ \ &[\lambda_0(\frac{k_0^2}{2} + ik_1\tilde{Y}_1 + 2ik_2\tilde{Y}_2) + \lambda_{-1}(k_1.k_0 + 2ik_2.\tilde{Y}_1) \\ &+\lambda_{-2}(2k_2.k_0 + \frac{1}{2}k_1.k_1) + \lambda_1(ik_1.\tilde{Y}_2 + ik_0.\tilde{Y}_1) \\ &+\lambda_2(-\frac{1}{2}\tilde{Y}_1.\tilde{Y}_1 + ik_0\tilde{Y}_2)]e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2}|0\rangle \end{aligned} \tag{8.1.112}$$

Since the vertex operators of interest have two k_1's or one k_2, the action of L_{+n} on them will give a term with two k_1's or one k_2. L_{+2} will give a term of level zero and multiplied by λ_2 and and L_{+1} will give a term of level one and multiplied by λ_1. To get gauge transformations L_{-1}, L_{-2} we need to extract the level two terms that have λ_{-1} and λ_{-2} respectively. We can write down terms quite easily:

[10]In [40] at level 3 the Q-rules were not used. Here we have used the Q-rules right from the beginning. Thus the discussion on Level 3 is modified a little.

1. $\lambda_0 \mathbf{L_0}$:

$$\lambda_0\ [\frac{k_0^2}{2}\ (-\frac{1}{2}k_{1\mu}k_{1\nu}\partial X^\mu \partial X^\nu + ik_{2\mu}\partial^2 X^\mu) - k_{1\mu}k_{1\nu}\partial X^\mu \partial X^\nu + 2ik_{2\mu}\partial^2 X^\mu]\ e^{ik_0.X}|0\rangle \tag{8.1.113}$$

2. $\lambda_{-1}\mathbf{L_1}$:

$$\lambda_{-1}[k_1.k_0\ ik_{1\mu}\partial X^\mu + 2ik_{2\mu}\partial X^\mu]\ e^{ik_0.X}|0\rangle \tag{8.1.114}$$

3. $\lambda_{-2}\mathbf{L_2}$:

$$\lambda_{-2}[2k_2.k_0 + \frac{1}{2}k_1.k_1]\ e^{ik_0.X}|0\rangle \tag{8.1.115}$$

4. $\lambda_1 \mathbf{L_{-1}}$:

$$\lambda_1[ik_{1\mu}\partial^2 X^\mu - k_{1\mu}k_{0\nu}\partial X^\mu \partial X^\nu]\ e^{ik_0.X}|0\rangle \tag{8.1.116}$$

(It is easy to see that the above is just $\lambda_1 L_{-1} ik_{1\mu}\partial X^\mu\ e^{ik_0.X}|0\rangle$)

5. $\lambda_2 \mathbf{L_{-2}}$:

$$\lambda_2[-\frac{1}{2}\partial X.\partial X + ik_{0\mu}\partial^2 X^\mu]\ e^{ik_0.X}|0\rangle \tag{8.1.117}$$

(This is just $\lambda_2 L_{-2} e^{ik_0.X}|0\rangle$)

The $L_0 = 1$ equation gives the mass shell condition and the requirements $L_1, L_2 V|0\rangle = 0$ give additional physical state constraints.

Since $L_n|0\rangle = 0, n \geq -1$, the constraints given above are equivalent to $[L_n, V] = 0, n \geq -1$. For the gauge transformations $L_{-2}|0\rangle \neq 0$. So $L_{-n}V|0\rangle = [L_{-n}, V]|0\rangle + Vl_{-n}|0\rangle$ and differs from the commutator. In LV formalism one does not include the second term viz. action on the vacuum. This has to be accounted for by field redefinitions.

Liouville Mode

One can obtain the physical state constraints, which are the action of L_{+n}, also by looking at the Liouville mode dependence. The Liouville mode,, ρ , is related to λ_n at linear order by

$$\frac{d\lambda}{dt} = \rho \tag{8.1.118}$$

where $\lambda(t) = \sum_n \lambda_n t^{1-n}$ and $\rho(t) = \rho(0) + t\partial\rho(0) + \frac{t^2}{2}\partial^2\rho(0) + ...$ Thus we get

$$\lambda_0 = \rho, \quad \lambda_{-1} = \frac{1}{2}\partial\rho, \quad \lambda_{-2} = \frac{1}{3!}\partial^2\rho, \quad \lambda_{-3} = \frac{1}{4!}\partial^3\rho \tag{8.1.119}$$

This way of looking at the constraints is useful for purposes of comparison with the LV formalism. Thus

$$e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2} =: e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2} :$$
$$e^{\frac{1}{2}[k_0^2\langle XX\rangle + 2k_1.k_0\langle X\partial X\rangle + k_1.k_1\langle \partial X\partial X\rangle + 2k_2.k_0\langle X\partial^2 X\rangle]}$$

$$=: e^{ik_0.\tilde{Y} + ik_1.\tilde{Y}_1 + ik_2.\tilde{Y}_2} : e^{\frac{1}{2}[k_0^2\rho + 2k_1.k_0\frac{1}{2}\partial\rho + k_1.k_1\frac{1}{6}\partial^2\rho + 2k_2.k_0\frac{1}{3}\partial^2\rho]} \tag{8.1.120}$$

The Liouville mode dependence is obtained using (8.1.119), (8.1.112). This implies $\langle XX\rangle = \rho$, $\langle X\partial X\rangle = \frac{1}{2}\rho$, $\langle X\partial^2 X\rangle = \frac{1}{3}\partial^2\rho$, $\langle \partial X\partial X\rangle = \frac{1}{6}\partial^2\rho$. These can be derived by other methods also [49].

In addition to the anomalous dependences, the Liouville mode also enters at the classical level. This can be obtained by writing covariant derivatives. The vertex operators on the boundary involve covariant derivatives ∇_x where x is the coordinate along the boundary of the world sheet. The vertex operators on

the boundary should be : $\int dxV$ where V is a one dimensional vector vertex operator or $\int dx\sqrt{g}S$ where S is one dimensional scalar. Note that $g_{xx} = g$ (in one dimension) and $g^{xx} = \frac{1}{g}$ The simplest vertex operator is thus $\nabla_x X = \partial_x X$ (since X is a scalar). Further $\nabla^x X = g^{xx}\nabla_x X$ and using $\nabla_x T^x = \frac{1}{\sqrt{g}}\partial_x(\sqrt{g}T^x)$ we get $\nabla_x\nabla^x X = \frac{1}{\sqrt{g}}\partial_x\sqrt{g}g^{xx}\partial_x X = \frac{1}{\sqrt{g}}\partial_x\frac{1}{\sqrt{g}}\partial_x X$ is a scalar. Thus $\sqrt{g}\nabla_x\nabla^x X = \partial_x\frac{1}{\sqrt{g}}\partial_x X$ is the vertex operator with two derivatives. One can similarly show that $\partial_x\frac{1}{\sqrt{g}}\partial_x\frac{1}{\sqrt{g}}\partial_x X$ is the vertex operator with three derivatives. This pattern continues.

The metric on the boundary induced by the metric on the bulk is:

$$g_{xx} = 2\frac{\partial z}{\partial x}\frac{\partial \bar{z}}{\partial x}g_{z\bar{z}} = g_{z\bar{z}} \tag{8.1.121}$$

Thus in conformal coordinates since $g_{z\bar{z}} = e^{-2\rho}$, we have $\frac{1}{\sqrt{g}} = e^{\rho}$. Thus

$$\int dx\ \partial X, \quad \int dx\ e^{\rho}(\partial^2 X + \partial\rho\partial X), \quad \int dx\ e^{2\rho}(\partial^3 X + 3\partial^2 X\partial\rho + \partial X\partial^2\rho), ... \tag{8.1.122}$$

Or if we remove $\int dx\sqrt{g}$ we get

$$e^{\rho}\partial X, \quad e^{2\rho}(\partial^2 X + \partial\rho\partial X), \quad e^{3\rho}(\partial^3 X + 3\partial^2 X\partial\rho + \partial X\partial^2\rho), ... \tag{8.1.123}$$

for the vertex operators. The power of e^{ρ} now counts the dimension of the unintegrated vertex operators. Inserting this into (8.1.137) we get:

$$=: e^{ik_0.X + ik_1.e^{\rho}\partial X + ik_2.e^{2\rho}(\partial^2 X + \partial\rho\partial X)} : e^{\frac{1}{2}[k_0^2\rho + 2k_1.k_0\frac{1}{2}\partial\rho + k_1.k_1\frac{1}{6}\partial^2\rho + 2k_2.k_0\frac{1}{3}\partial^2\rho]} \tag{8.1.124}$$

This expression gives the complete ρ dependence to linear order. The coefficient of $\lambda_{-1} = \frac{1}{2}\partial\rho$ is $2ik_{2\mu}\partial X^{\mu} + k_1.k_0 ik_{1\mu}\partial X^{\mu}$ and that of $\lambda_{-2} = \frac{1}{6}\partial^2\rho$ is $(\frac{1}{2}k_1.k_1 + 2k_2.k_0)$ as required.

Space-time Fields

We can define fields as usual [23, 16] by replacing $k_{1\mu}k_{1\nu}$ by $\Phi_{\mu\nu}$ and $k_{2\mu}$ by A_{μ}. Thus the level two boundary action is

$$\int dx\ [-\frac{1}{2}\Phi_{\mu\nu}\partial X^{\mu}\partial X^{\nu} + iA_{\mu}\partial^2 X^{\mu}]$$

The gauge parameters are obtained by replacing $\lambda_1 k_{1\mu}$ by ϵ^{μ} and λ_2 by ϵ_2. Then we have the following:

Constraints: The mass shell constraint fixes $p^2 + 2 = 0$. In addition we have,

1. $$p_{\nu}\Phi^{\nu\mu} + 2A^{\mu} = 0 \tag{8.1.125}$$

2. $$\Phi^{\nu}_{\nu} + 4p_{\nu}A^{\nu} = 0 \tag{8.1.126}$$

Gauge transformations:

$$\delta\Phi^{\mu\nu} = \eta^{\mu\nu}\epsilon_2 + p^{(\mu}\epsilon^{\nu)}$$

$$\delta A^{\mu} = p^{\mu}\epsilon_2 + \epsilon^{\mu} \tag{8.1.127}$$

On mass shell ($p^2 + 2 = 0$), if we set $\epsilon_2 = 0$ and $p^{\mu}\epsilon_{\mu} = 0$ then the constraints are invariant under the gauge transformations. This symmetry (which corresponds to L_{-1}) allows us to gauge away all transverse A_{μ} that satisfy $p.A = 0$. The constraint then says that the longitudinal part is equal to the trace of Φ. Furthermore there is an additional symmetry: the constraints are invariant under the gauge transformations with $\epsilon^{\mu} = \frac{3}{2}p^{\mu}\epsilon_2$, along with the mass shell condition $p^2 + 2 = 0$ and the critical dimension $D = 26$. Both these symmetries transformations correspond to adding zero norm states: states that are physical as well as

pure gauge. The second one allows us to remove the trace of Φ or equivalently the longitudinal component of A_μ as well. This second symmetry thus reduces the number of physical degrees by one again. The net effect is to remove two polarizations, which is why the light cone gauge description with D-2 polarizations is possible. It is easy to see that this second gauge transformation corresponds to the state $(L_{-2}+\frac{3}{2}L_{-1}^2)e^{ik_0.X}|0\rangle$. Finally, when all the dust settles, we are left with one symmetric, traceless, transverse 2-tensor, which are described by the light cone oscillators as described earlier in Section 4.

8.1.2 Level 3

Vertex Operators

The vertex operators in the OC formalism at this level can be written down as follows:

$$e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3\tilde{Y}_3}|0\rangle =$$

$$(...+i\frac{k_3^\mu}{2!}\partial^3X^\mu-k_{2\mu}k_{1\nu}\partial^2X^\mu\partial X^\nu-i\frac{k_{1\mu}k_{1\nu}k_1^\rho}{3!}\partial X^\mu\partial X^\nu\partial X^\rho+...)e^{ik_0.X}|0\rangle \tag{8.1.128}$$

Action of $L_{\pm n}$

Using the same equation (8.1.110) one gets:

$$\begin{aligned} exp\ \ [\lambda_3(ik_0.\tilde{Y}_3-\tilde{Y}_1.\tilde{Y}_2)+\lambda_2(ik_1\tilde{Y}_3+ik_0\tilde{Y}_2-\frac{\tilde{Y}_1.\tilde{Y}_1}{2})+\lambda_1(i2k_2\tilde{Y}_3+ik_1\tilde{Y}_2+ik_0\tilde{Y}_1) \\ +\lambda_0(i3k_3\tilde{Y}_3+2ik_2\tilde{Y}_2+ik_1\tilde{Y}_1+\frac{k_0^2}{2})+\lambda_{-1}(i3k_3\tilde{Y}_2+i2k_2\tilde{Y}_1+k_1.k_0) \\ +\lambda_{-2}(i3k_3\tilde{Y}_1+2k_2.k_0+\frac{k_1.k_1}{2})+\lambda_{-3}(3k_3.k_0+2k_2.k_1)]e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3\tilde{Y}_3}|0\rangle \end{aligned} \tag{8.1.129}$$

One can extract as before the action of L_{+n} on the vertex operators of dimension three by extracting terms of dimension $3-n$ involving k_3, k_2k_1 and $k_1k_1k_1$ in (8.1.129). Similarly gauge transformations corresponding to L_{-n} are obtained by extracting the level three terms that have λ_n in (8.1.129).

1. $\lambda_\mathbf{0}\mathbf{L_0}$:

$$\lambda_0(3+\frac{k_0^2}{2})[ik_{3\mu}\tilde{Y}_3^\mu-k_{2\mu}k_{1\nu}\tilde{Y}_2^\mu\tilde{Y}_1^\nu-ik_{1\mu}k_{1\nu}k_{1\rho}\tilde{Y}_1^\mu\tilde{Y}_1^\nu\tilde{Y}_1^\rho] \tag{8.1.130}$$

2. $\lambda_{-\mathbf{1}}\mathbf{L_1}$:

$$\lambda_{-1}[(i3k_{3\mu}+ik_{2\mu}k_1.k_0)\tilde{Y}_2^\mu-(2k_{2\mu}k_{1\nu}+\frac{k_{1\mu}k_{1\nu}}{2}k_1.k_0)\tilde{Y}_1^\mu\tilde{Y}_1^\nu] \tag{8.1.131}$$

3. $\lambda_{-\mathbf{2}}\mathbf{L_2}$:

$$\lambda_{-2}[i3k_{3\mu}\tilde{Y}_1^\mu+2k_2.k_0ik_{1\mu}\tilde{Y}_1^\mu+\frac{k_1.k_1}{2}ik_{1\mu}\tilde{Y}_1^\mu] \tag{8.1.132}$$

4. $\lambda_{-\mathbf{3}}\mathbf{L_3}$:

$$\lambda_{-3}[3k_0.k_3+2k_2.k_1] \tag{8.1.133}$$

5. $\lambda_\mathbf{1}\mathbf{L_{-1}}$:

$$\lambda_1[i2k_{2\mu}\tilde{Y}_3^\mu-k_{1\mu}k_{1\nu}\tilde{Y}_2^\mu\tilde{Y}_1^\nu-k_{0\mu}k_{2\nu}\tilde{Y}_1^\mu\tilde{Y}_2^\nu+\frac{i}{2}k_{0\mu}k_{1\nu}k_{1\rho}\tilde{Y}_1^\mu\tilde{Y}_1^\nu\tilde{Y}_1^\rho] \tag{8.1.134}$$

6. $\lambda_2 \mathbf{L_{-2}}$:

$$\lambda_2[ik_{1\mu}\tilde{Y}_3^\mu - k_{0\mu}k_{1\nu}\tilde{Y}_2^\mu\tilde{Y}_1^\nu - ik_{1\mu}\tilde{Y}_1^\mu\frac{\tilde{Y}_1.\tilde{Y}_1}{2}] \tag{8.1.135}$$

7. $\lambda_3 \mathbf{L_{-3}}$:

$$\lambda_3[ik_{0\mu}\tilde{Y}_3^\mu - \tilde{Y}_1.\tilde{Y}_2] \tag{8.1.136}$$

Liouville Mode

Exactly as in the level two case one can get the Liouville mode dependences - both the classical and anomalous terms.

$$e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3.\tilde{Y}_3} =: e^{ik_0.\tilde{Y}+ik_1.\tilde{Y}_1+ik_2.\tilde{Y}_2+ik_3.\tilde{Y}_3} :$$

$$e^{\frac{1}{2}[k_0^2\langle XX\rangle+2k_1.k_0\langle X\partial X\rangle+k_1.k_1\langle\partial X\partial X\rangle+2k_2.k_0\langle X\partial^2 X\rangle+2k_3.k_0\langle\frac{\partial^3 X}{2!}X\rangle+2k_2.k_1\langle\partial^2 X\partial X\rangle]}$$

This is the anomalous dependence. Using covariant derivatives gives the classical part also:

$$=: e^{ik_0.X+ik_1.e^\rho\partial X+ik_2.e^{2\rho}(\partial^2 X+\partial\rho\partial X)+i\frac{1}{2}k_3.e^{3\rho}(\partial^3 X+3\partial^2 X\partial\rho+\partial X\partial^2\rho)} :$$

$$e^{\frac{1}{2}[k_0^2\rho+2k_1.k_0\frac{1}{2}\partial\rho+k_1.k_1\frac{1}{6}\partial^2\rho+2k_2.k_0\frac{1}{3}\partial^2\rho+2k_3.k_0\frac{\partial^3\rho}{8}+2k_2.k_1\frac{\partial^3\rho}{12}]} \tag{8.1.137}$$

We have used $\langle\partial^3 XX\rangle = \frac{\partial^3\rho}{4}$ and $\langle\partial^2 X\partial X\rangle = \frac{\partial^3\rho}{12}$. Using (8.1.119) giving the relation between λ_n and $\partial^n\rho$ one can check that this is the same as the results given above. This form is useful for comparison with the loop variable formalism where analogous terms are present.

Space-time Fields

We introduce space-time fields as before by replacing $k_{1\mu}k_{1\nu}k_{1\rho}$ by $\Phi^{\mu\nu\rho}$, $k_{2\mu}k_{1\nu}$ by $B^{\mu\nu}+C^{\mu\nu}$ where B is symmetric and C is antisymmetric, and k_3^μ by A^μ. Thus the boundary action is

$$\int dx\ [-\frac{i}{3!}\Phi_{\mu\nu\rho}\tilde{Y}_1^\mu\tilde{Y}_1^\nu\tilde{Y}_1^\rho - (B_{\mu\nu}+C_{\mu\nu})\tilde{Y}_2^\mu\tilde{Y}_1^\nu + iA_3\tilde{Y}_3^\mu]e^{ik_0.\tilde{Y}}$$

For the gauge parameters we let λ_3 be ϵ_3, $\lambda_2 k_{1\mu}$ be ϵ_{12}^μ, $\lambda_1 k_{2\mu}$ be ϵ_{21}^μ and $\lambda_1 k_{1\mu}k_{1\nu}$ be $\epsilon_{111}^{\mu\nu}$. We then have:

Constraints:

The mass shell constraint $L_0 = 1$ becomes $p^2+4=0$. In addition,

1. L_1

$$p^\nu(B_{\mu\nu}+C_{\mu\nu})+3A_\mu = 0 \tag{8.1.138}$$

2. L_1

$$\frac{p^\rho\Phi_{\rho\mu\nu}}{4}+B_{\mu\nu} = 0 \tag{8.1.139}$$

3. L_3

$$\frac{B^\nu{}_\nu}{12}+\frac{p^\nu A_\nu}{8} = 0 \tag{8.1.140}$$

4. L_2

$$\frac{\Phi^\nu{}_{\nu\mu}}{12}+\frac{A_\mu}{2}+\frac{p^\nu(B_{\nu\mu}+C_{\nu\mu})}{3} = 0 \tag{8.1.141}$$

Gauge Transformations:

$$\delta\Phi^{\mu\nu\rho} = \epsilon_{12}^{(\mu}\eta^{\nu\rho)} + p^{(\mu}\epsilon_{111}^{\nu\rho)}$$
$$\delta(B^{\mu\nu} + C^{\mu\nu}) = p^{\mu}\epsilon_{12}^{\nu} + p^{\nu}\epsilon_{21}^{\mu} + \epsilon_{111}^{\mu\nu} + \epsilon_3\eta^{\mu\nu}$$
$$\delta A^{\mu} = p^{\mu}\epsilon_3 + \epsilon_{12}^{\mu} + 2\epsilon_{21}^{\mu} \tag{8.1.142}$$

(Symmetrization indicated in the first line involves adding two other orderings giving three permutations for each term.)

Invariance of Constraints under Gauge Transformations:

One can show that the constraints are invariant under the gauge transformation under some conditions. Invariance of constraint 1 imposes the following condition on the gauge parameters using:

$$\begin{aligned} p.\epsilon_{12} + 4\epsilon_3 &= 0 \\ p^{\nu}\epsilon_{111\mu\nu} + 3\epsilon_{12\mu} + (p^2+6)\epsilon_{21\mu} &= 0 \end{aligned} \tag{8.1.143}$$

Invariance of constraint 2 requires:

$$\begin{aligned} p^2 + 4 &= 0 \\ p^{\nu}\epsilon_{111\mu\nu} + 3\epsilon_{12\mu} + 2\epsilon_{21\mu} &= 0 \\ p.\epsilon_{12} + 4\epsilon_3 &= 0 \end{aligned} \tag{8.1.144}$$

Clearly for $p^2 = -4$ the two sets (8.1.143) and (8.1.144) are equivalent.

Invariance of constraint 3 requires (using the above conditions) the following additional condition between gauge parameters.

$$\frac{2D + 3p^2 - 20}{24}\epsilon_3 + \frac{8(p.\epsilon_{21})}{24} + \frac{\epsilon^{\nu}{}_{\nu}}{24} = 0 \tag{8.1.145}$$

Finally using the above conditions and setting $D = 26$ and $p^2 + 4 = 0$ one finds that the fourth constraint is satisfied.

Using $\epsilon_{111\mu\nu}$ one can gauge away $B_{\mu\nu}$, and using one of the vector parameters $\epsilon_{12\mu}$ one can gauge away A_{μ}. (The second vector gauge parameter is then fixed by the conditions (8.1.143).) Then constraint 1 says that $C_{\mu\nu}$ is transverse and constraint 2 says that $\Phi_{\mu\nu\rho}$ is transverse. Using these we also see from constraint 4 that Φ is traceless. Thus we have a traceless and transverse symmetric two tensor and an antisymmetric two tensor which is the correct count for level 3 open strings.

8.2 Loop Variable Formalism

We now consider the system of constraints and gauge transformations in the loop variable approach. The generalized Liouville mode (Σ) dependence is given in (3.2.17) in Section 3.2.

We consider the level two and three operators in turn.

8.2.1 Level 2

Vertex Operators

$$e^{\frac{(k_0^2+q_0^2)}{2}\Sigma}[ik_{2\mu}Y_2^{\mu} + iq_2\theta_2 - \frac{k_{1\mu}k_{1\nu}}{2}Y_1^{\mu}Y_1^{\nu} - \frac{q_1q_1}{2}\theta_1\theta_1 - k_{1\mu}q_1Y_1^{\mu}\theta_1$$
$$+i(k_{1\mu}Y_1^{\mu} + q_1\theta_1)(k_1.k_0 + q_1q_0)\frac{1}{2}\frac{\partial\Sigma}{\partial x_1} + (k_2.k_0 + q_2q_0)\frac{1}{2}\frac{\partial\Sigma}{\partial x_2} +$$
$$-i\frac{k_{1\mu}k_{1\nu}}{2}q_1Y_1^{\mu}Y_1^{\nu}\theta_1 - i\frac{q_1q_1}{2}k_{1\mu}\theta_1\theta_1Y_1^{\mu}$$

$$\frac{(k_1.k_1+q_1q_1)}{2}\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1^2}-\frac{\partial\Sigma}{\partial x_2})]e^{ik_0.Y} \tag{8.2.146}$$

We have written the dimensionally reduced version. Weyl Invariance is independence of Σ. The coefficients of Σ and its derivatives have to be set to zero. There are the constraints. It will be seen that field redefinitions will make these equivalent to the constraints of the OC formalism (8.1.125) and (8.1.126). This implies that the classical Liouville mode dependence is included here indirectly through terms involving q_n.

Space-time Fields, Gauge Transformations and Constraints

- **Space-time Fields:**

  The fields are obtained by setting $k_{1\mu}k_{1\nu}\approx S_{11}^{\mu\nu}$, $k_{2\mu}\approx S_2^\mu$, $k_{1\mu}q_1q_0\approx S_{11}^\mu$, $q_1q_1\approx S_{11}$, and $q_2q_0\approx S_2$. The gauge parameters are $\lambda_2\approx\Lambda_2$, $\lambda_1k_{1\mu}\approx\Lambda_{11}^\mu$, $\lambda_1q_1q_0\approx\Lambda_{11}$.

- **Gauge Transformations:**

$$\delta S_{11}^{\mu\nu}=p^{(\mu}\Lambda_{11}^{\nu)},\quad \delta S_2^\mu=\Lambda_{11}^\mu+p^\mu\Lambda_2,\quad \delta S_{11}^\mu=\Lambda_{11}^\mu+p^\mu\Lambda_{11},$$

$$\delta S_2=\Lambda_2q_0^2+\Lambda_{11},\quad \delta S_{11}=2\Lambda_{11} \tag{8.2.147}$$

  Now one can make the following identifications: $q_1q_1\approx q_2q_0$, $k_{1\mu}q_1\approx k_{2\mu}q_0$, and $\lambda_1q_1\approx\lambda_2q_0$. This gives: $S_{11}^\mu=2S_2^\mu$, $S_{11}\approx S_2$ and $\Lambda_{11}\approx 2\Lambda_2$ and the gauge transformations are consistent with these identifications.

- **Constraints:**

  The coefficient of Σ gives the usual mass shell condition $p^2+q_0^2=0$. Note that the $(mass)^2$ equals the dimension of the operator, but the Σ dependence representing this (and also all other Σ dependences) comes from an anomaly rather than from the classical dependence as in the OC formalism.

  1. Coefficient of $\frac{\partial^2\Sigma}{\partial x_1^2}$
$$k_1.k_1+q_1q_1=0\quad\Rightarrow S_{11\mu}^\mu+S_2=0 \tag{8.2.148}$$
  2. Coefficient of $\frac{\partial\Sigma}{\partial x_2}$
$$k_2.k_0+q_2q_0=0\quad\Rightarrow p_\mu S_2^\mu+S_2=0 \tag{8.2.149}$$
  3. Coefficient of $\frac{\partial\Sigma}{\partial x_1}Y_1^\mu$
$$(k_1.k_0+q_1q_0)k_{1\mu}=0\quad\Rightarrow p_\nu S_{11}^{\mu\nu}+2S_2^\mu=0 \tag{8.2.150}$$

  The constraint proportional to θ_1 is seen to be a linear combination of the above. The equations of motion are obtained by setting the variational derivative of Σ equal to zero, and are gauge invariant.

8.2.2 Level 3

Vertex Operator The complete Level 3 gauge covariantized vertex operator is:

$$e^{\frac{(k_0^2+q_0^2)}{2}\Sigma}[ik_3^\mu Y_3^\mu+iq_3\theta_3-k_{2\mu}k_{1\nu}Y_2^\mu Y_1^\nu-$$
$$q_1q_2\theta_1\theta_2-k_{1\mu}q_2Y_1^\mu\theta_2-k_{2\mu}q_1Y_2^\mu\theta_1-i\frac{k_{1\mu}k_{1\nu}k_1^\rho}{3!}Y_1^\mu Y_1^\nu Y_1^\rho-i\frac{(q_1)^3}{3!}(\theta_1)^3$$
$$+i(k_{2\mu}Y_2^\mu+q_2\theta_2)(k_1.k_0+q_1q_0)\frac{1}{2}\frac{\partial\Sigma}{\partial x_1}$$

$$+i(k_{1\mu}Y_1^\mu + q_1\theta_1)[(k_2.k_0 + q_2q_0)\frac{1}{2}\frac{\partial\Sigma}{\partial x_2} + \frac{1}{2}(k_1.k_1 + q_1q_1)\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1\partial x_1} - \frac{\partial\Sigma}{\partial x_2})]$$

$$+(k_3.k_0 + q_3q_0)\frac{1}{2}\frac{\partial\Sigma}{\partial x_3} + (k_2.k_1 + q_2q_1)\frac{1}{2}(\frac{\partial^2\Sigma}{\partial x_1\partial x_2} - \frac{\partial\Sigma}{\partial x_3})]e^{ik_0.Y} \tag{8.2.151}$$

Space-time Fields, Gauge Transformations and Constraints

- **Space-time Fields**

  This was worked out in Section 4 (8.2.152) and we reproduce it here for convenience:

$$\begin{aligned}
\langle k_{1\mu}k_{1\nu}k_{1\rho}\rangle &= S_{111\mu\nu\rho}\\
\langle k_{2[\mu}k_{1\nu]}\rangle &= A_{21[\mu\nu]}\\
\langle k_{2(\mu}k_{1\nu)} = k_{1\mu}k_{1\nu}q_1\rangle &= S_{21(\mu\nu)}\\
\langle k_{3\mu}q_0^2 = k_{1\mu}q_1^2 = \frac{1}{2}(k_{1\mu}q_2 + k_{2\mu}q_1)q_0\rangle &= S_{3\mu}q_0^2\\
\langle k_{1\mu}q_2\rangle &= S_{12\mu}\\
\langle q_3q_0^2 = q_2q_1q_0 = q_1^3\rangle &= S_3q_0^2
\end{aligned} \tag{8.2.152}$$

- **Gauge Parameters**

$$\begin{aligned}
\langle\lambda_1q_1q_1 = \frac{1}{2}(\lambda_2q_1 + \lambda_1q_2)q_0 = \lambda_3q_0^2\rangle &= \Lambda_3q_0^2\\
\langle\frac{1}{2}(\lambda_1q_2 - \lambda_2q_1)\rangle &= \Lambda_Aq_0\\
\langle\lambda_1q_1k_{1\mu} = \frac{1}{2}(\lambda_2k_{1\mu} + \lambda_1k_{2\mu})q_0\rangle &= \frac{1}{2}(\Lambda_{12\mu} + \Lambda_{21\mu})q_0 = q_0\Lambda_{S\mu}\\
\langle\frac{1}{2}(\lambda_2k_{1\mu} - \lambda_1k_{2\mu})\rangle &= \frac{1}{2}(\Lambda_{21\mu} - \Lambda_{12\mu}) = \Lambda_{A\mu}\\
\langle\lambda_1k_{1\mu}k_{1\nu}\rangle &= \Lambda_{111\mu\nu}
\end{aligned} \tag{8.2.153}$$

  Note that there is also a tracelessness condition:

$$\Lambda_{111\ \mu}^{\ \ \ \ \mu} + \Lambda_3q_0^2 = 0 \tag{8.2.154}$$

- **Gauge Transformations**

$$\begin{aligned}
\delta S_3 &= 3\Lambda_3q_0\\
\delta S_{3\mu} &= 2\Lambda_{S\mu} + k_{0\mu}\Lambda_3\\
\delta S_{12\mu} &= q_0[2\Lambda_{S\mu} + \Lambda_{A\mu} + (\Lambda_A + \Lambda_3)k_{0\mu}]\\
\delta(S_{12\mu} - S_{3\mu}q_0) \equiv \delta S_{A\mu}q_0 &= \Lambda_{A\mu}q_0 + k_{0\mu}\Lambda_Aq_0\\
\delta S_{\mu\nu} &= \Lambda_{111\mu\nu} + k_{0(\mu}\Lambda_{S\nu)}\\
\delta A_{\mu\nu} &= k_{0[\mu}\Lambda_{A\nu]}\\
\delta S_{111\mu\nu\rho} &= k_{0(\mu}\Lambda_{111\nu\rho)}
\end{aligned} \tag{8.2.155}$$

- **Constraints**

  Level 3 terms involving Σ derivatives in (8.2.151) give the constraints. In writing them below, the Q-rules (4.1.20) have been used.

  1. $$k_3.k_0 + q_3q_0 = 0 \quad \Rightarrow p^\nu S_{3\nu} + S_3q_0 = 0 \tag{8.2.156}$$

2.

$$k_2.k_1 + q_2 q_1 = 0 \quad \Rightarrow S^{\mu}_{21\mu} + S_3 q_0 = 0 \tag{8.2.157}$$

3.

$$(k_1.k_1 + q_1 q_1)k_{1\mu} = 0 \quad \Rightarrow S^{\nu}_{111\mu\nu} + S_{3\mu} q_0^2 = 0 \tag{8.2.158}$$

4.

$$(k_2.k_0 + q_2 q_0)k_{1\mu} = 0 \quad \Rightarrow p^{\nu}(S_{21\nu\mu} + A_{21\nu\mu}) + S_{12\mu} q_0 = 0$$

$$\Rightarrow p^{\nu}(S_{21\nu\mu} + A_{21\nu\mu}) + (S_{3\mu} + S_{A\mu}) q_0 = 0 \tag{8.2.159}$$

5.

$$(k_1.k_0 + q_1 q_0)k_{1\mu} k_{1\nu} = 0 \quad \Rightarrow p^{\rho} S_{111\mu\nu\rho} + q_0^2 S_{21\mu\nu} = 0 \tag{8.2.160}$$

6.

$$(k_1.k_0 + q_1.q_0)k_{2\mu} = 0 \quad \Rightarrow p^{\nu}(S_{21\mu\nu} + A_{21\mu\nu}) + 2S_{3\mu} q_0^2 - S_{12\mu} q_0 = 0$$

$$\Rightarrow p^{\nu}(S_{21\nu\mu} - A_{21\nu\mu}) + (S_{3\mu} - S_{A\mu}) q_0^2 = 0 \tag{8.2.161}$$

7.

$$(k_2.k_0 + q_2 q_0)q_1 q_0 = 0 \quad \Rightarrow p^{\nu}(2S_{3\nu} q_0 - S_{12\nu}) + S_3 q_0^2 = 0$$

$$\Rightarrow p^{\nu}(S_{3\nu} - S_{A\nu}) q_0 + S_3 q_0^2 = 0 \tag{8.2.162}$$

Note that the last constraint comes from $\frac{\partial \Sigma}{\partial x_2}\theta_1$. Combining this with the first constraint gives $p^{\nu} S_{A\nu} = 0$. This is not a new constraint: Combining constraints 4 and 6 we get using the antisymmetry of $A_{21\mu\nu}$:

$$p^{\nu} A_{21\nu\mu} + S_{A\mu} = 0 \ \Rightarrow p^{\nu} p^{\mu} A_{21\nu\mu} + p^{\mu} S_{A\mu} \ \Rightarrow p^{\mu} S_{A\mu} = 0$$

Constraint 6 says that only one of the vectors is independent. Constraint 1 says the scalar is not independent. If we remove one vector and one scalar we have the same number of fields as in the OC formalism.

- **Invariance of Constraints Under Gauge Transformations:** It can be checked that the constraints are invariant under the gauge transformations provided some transversality constraints are satisfied by the gauge parameter. However there is no constraint on D. The mass shell constraint becomes $p^2 + q_0^2 = 0$ with arbitrary q_0. One finds the following conditions.

$$\begin{aligned} p.\Lambda_S + \Lambda_3 q_0^2 &= 0 \\ k_0^{\nu} \Lambda_{111\mu\nu} + \Lambda_{S\mu} q_0^2 &= 0 \\ \Lambda_{111}{}^{\nu}{}_{\nu} + \Lambda_3 q_0^2 &= 0 \\ p^{\nu} \Lambda_{A\nu} + \Lambda_A q_0^2 &= 0 \end{aligned} \tag{8.2.163}$$

The third condition on $\Lambda_{111\mu\nu}$ is nothing but the D+1 dimensional tracelessness constraint on the gauge parameter which is always there - even for gauge invariance of the equations. The other three can be understood as generalized (i.e in D+1 dimensions) transversality. Thus the conditions below are equivalent to the ones above if one uses the Q-rules.

$$p^{\nu} \lambda_1 k_{1\mu} k_{1\nu} = 0 \ ; \quad p^{\mu} \lambda_1 k_{2\mu} = 0 \quad ; \ p^{\mu} \lambda_2 k_{1\mu} = 0 \ ; \quad \lambda_1 k_1 \circ k_1 = 0 \ ; \ \mu : 0 \ - \ D$$

Here as before $\circ$ denotes a D+1 dimensional dot product.

One can gauge away $S_{21\mu\nu}$ and this automatically ensures the transversality of $S_{111\mu\nu\rho}$ by the constraint 5. Gauging away $S_3\mu$ also gets rid of the trace of $S_{111\mu\nu\rho}$ by constraint 3. It also gets rid of S_3 by constraint 1. Thus we have a transverse traceless 3 tensor as required. The antisymmetric 2 tensor is transverse once $S_{A\mu}$ is gauge away. This completes the counting of degrees of freedom.

8.3 Mapping from OC Formalism to LV Formalism

8.3.1 Level 2

Mapping of fields

The mapping is given by:

$$\Phi^{\mu\nu} = S_{11}^{\mu\nu} + (\eta^{\mu\nu} + \frac{5}{2}p^\mu p^\nu)S_2$$

$$A^\mu = S_2^\mu + 2p^\mu S_2 \tag{8.3.164}$$

Mapping Gauge Transformations

If one makes a LV gauge transformation with parameters Λ_{11}^μ and Λ_2 one obtains:

$$\delta\Phi^{\mu\nu} = p^{(\mu}\Lambda_{11}^{\nu)} + (\eta^{\mu\nu} + \frac{5}{2}p^\mu p^\nu)4\Lambda_2$$

$$\delta A^\mu = \Lambda_{11}^\mu + 9\ p^\mu \Lambda_2 \tag{8.3.165}$$

The relative values of the different terms in (8.3.164), and therefore in the gauge transformation (8.3.165), are fixed by requiring that the gauge transformation be generated by some combination of L_{-n}'s. One can check that (8.3.165) corresponds to a gauge transformation by $4\Lambda_2(L_{-2} + \frac{5}{4}L_{-1}^2) + \Lambda_{11}^\mu L_{-1}$.

Mapping Constraints

Now consider the constraint (8.1.126). We see that

$$p_\nu \Phi^{\nu\mu} + 2A^\mu = p_\nu S_{11}^{\mu\nu} + 2S_2^\mu + (5 - \frac{5}{2}q_0^2)p^\mu S_2 \tag{8.3.166}$$

Only for $q_0^2 = 2$ does it become the LV constraint $p_\nu S_{11}^{\mu\nu} + 2S_2^\mu$. Furthermore

$$4p.A + \Phi^\mu_\mu = 4p.S_2 + S^\mu_{11\mu} + [(D - \frac{5}{2}q_0^2) - 8q_0^2]S_2 \tag{8.3.167}$$

This should equal

$$4(p.S_2 + S_2) + S^\mu_{11\mu} + S_2 \tag{8.3.168}$$

This fixes $D = 26$ (using $q_0^2 = 2$). Thus we see that the while the LV equations are gauge invariant in any dimension, when we require equivalence with OC formalism the critical dimension is picked out.

Equivalence of OC and LV Formalisms

Now we can see that the LV formalism is equivalent to the OC formalism: Start with a vertex operator in the OC formalism with fields that obey (8.3.166, 8.3.167). This implies that the corresponding LV vertex operator obeys the same constraint (8.3.166). Similarly (8.3.167) implies (8.3.168). This is the sum of two constraints of the LV formalism. Since the LV formalism is gauge invariant one can choose a gauge (using invariance under Λ_2 transformations), where S_2 to equal $-p.S_2$. This implies (by the constraint) that $S^\mu_{11\mu} + S_2 = 0$. Thus if the fields obey the physical state constraints of the OC formalism, then using the gauge invariance, we see that the LV constraints are also satisfied. In the reverse direction it is easier because we just have to take a linear combination of two LV constraints (8.2.148-8.2.150) to get an OC constraint.

We can go further in analyzing the constraints. After obtaining $p.S_2 + S_2 = 0$ using a Λ_2 transformations, there is a further invariance involving both Λ_{11}^μ and Λ_2 with $p.\Lambda_{11} + q_0^2\Lambda_2 = 0$. This transformation preserves all the constraints. (We also have to use the mass shell condition $p^2 + q_0^2 = 0$.) Using this invariance we can set $S_2 = 0$ while preserving $p.S_2 + S_2 = 0 = S^\mu_{11\mu} + S_2$. This then implies that $p.S_2 = S^\mu_{11\mu} = 0$. A very similar analysis done on the OC side using the constraint $4p.A + \Phi^\mu_\mu = 0$ and the gauge transformations with $\epsilon^\mu + \frac{3}{2}p^\mu\epsilon$ that preserves the constraint (provided $D = 26$, $q_0^2 = 2$): we can use it to set $p.A$ to zero and so $\Phi^\mu_\mu = 0$. Thus on both sides we have a transverse vector and a traceless tensor obeying $p_\nu\Phi^{\nu\mu} + 2A^\mu = 0$.

There are also terms involving θ_n on the LV side. But if we focus on the equations of motion involving only vertex operators with Y_n, then the two systems are identical. Since the physical states of the string are

conjugate to these vertex operators (that have only Y_n), this is all we need to describe the physics of string theory.

Finally we can use transverse gauge transformations involving ϵ^μ (i.e with $p.\epsilon = 0$) to gauge away the transverse vector A^μ (and the same thing can be done on the LV side to gauge away S_2^μ). This leaves a tensor $\Phi^{\mu\nu}$ which is transverse - $p_\nu \Phi^{\nu\mu} = 0$ - and traceless. This is the right number of degrees of freedom for the first massive state of the bosonic open string.

This concludes the demonstration of the equivalence of the OC formalism and LV formalism for Level 2 at the free level.

8.3.2 Level 3

We have seen in earlier subsections that in both formalisms the physical degrees of freedom are the same as that of the open string at level 3. It is thus clear that there exists a map from one set of fields to the other. We have also seen that the OC formalism requires D=26 and $p^2 = -4$, whereas the LV formalism is valid in any dimension and any q_0^2. Thus one expects that the map from one to the other, with both constraints and gauge transformations being mapped respectively, will work only in $D = 26$ and for $q_0^2 = 4$. We saw this explicitly in the last section for level 2. Level 3 is more complicated. We work out some details - just enough to see the critical dimension emerging.

We start by defining a fairly general map between the fields of the OC and LV formalisms:

$$\begin{aligned}
\Phi_{\mu\nu\rho} &= f_1 S_{111\mu\nu\rho} + f_2 S_{3(\mu}\eta_{\nu\rho)} + f_3 S_{A(\mu}\eta_{\nu\rho)} + f_4 p_{(\mu} S_{\nu\rho)} \\
B_{\mu\nu} &= b_1 S_{\mu\nu} + b_2 p_{(\mu} S_{3\nu)} + b_3 p_{(\mu} S_{A\nu)} + b_4 S_3 \eta_{\mu\nu} \\
C_{\mu\nu} &= c_1 A_{\mu\nu} + c_2 p_{[\mu} S_{3\nu]} + c_3 p_{[\mu} S_{A\nu]} \\
A_\mu &= a_1 S_{3\mu} + a_2 S_{A\mu} + a_3 p_\mu S_3
\end{aligned} \tag{8.3.169}$$

This is not the most general map - derivatives of the scalar field can be added in some places. We now write down the constraints in the OC formalism and require that they be satisfied when the LV constraints are used. This gives us linear equations in the coefficients. These equations also involve D and q_0. This is not enough to fix D or q_0. We then require that the gauge transformations can be consistently mapped from one to the other (i.e. that a consistent map between the gauge parameters should exist). This gives several more equations. We use the tensor and vector gauge parameters in the analysis. The scalar parameters are not included in this analysis. The resultant equations and solution are given in Appendix D (E). We see that D and q_0 get fixed exactly as in the level 2 case. We have not attempted to fix all the coefficients - it is quite tedious and not very illuminating for our purposes.

Thus in this section we have shown that at Level 2 and 3, the LV formalism and OC formalism describe the same degrees of freedom once we analyze the constraints and gauge transformations. For level 2 we also gave an explicit map of the fields. The map works only in the critical dimension and with the correct string spectrum. We expect this to continue to be true at higher levels also and also for closed strings.

Once the theory is gauge fixed and constraints imposed, the vertex operators are identical in the two theories. Thus the S-matrix is also the same. The theories describe the same physics. Thus we conclude that the gauge invariant ERG equations obtained here must describe string theory in the critical dimension. In other dimensions the theory described by the LV formalism is gauge invariant and hence plausibly consistent classically, but may or may not be equivalent to a non critical string theory. Quantum mechanical consistency is an open question.

Note that the loop variable formalism gives results right away in a convenient form - as a massless theory dimensionally reduced. Neither the OC formalism nor the BRST formalism does this. The Q-rules have to be imposed to get the spectrum in the LV formalism to match that of strings. These rules are also consistent with dimensional reduction. The higher dimensional origin of this theory is intriguing. It is tempting to speculate that this is related to M-theory.

9 Conclusions and Open Questions

The loop variable approach is an attempt to construct a formalism where all the gauge symmetries of string theory are present.The gauge symmetry associated with the massless graviton is general coordinate invariance. Making this manifest as done in standard General Relativity makes the formalism manifestly background independent. As a by product it gives us gauge invariant and generally covariant (interacting) equations of motion for massive fields in an arbitrary gravitational background.

The main tool used is the "loop variable" - which has all the vertex operators of the string collected into one non local loop variable. The equations are the Exact Renormalization Group equations for the word sheet theory, which impose conformal invariance. An infinite number of "proper time" variables are present in this formalism which allows us to work in the space of all gauge transformations. This makes the equations gauge invariant. The main guiding principles are world sheet conformal invariance and space time gauge invariance. World sheet reparametrization invariance is not imposed.

The extra coordinate that has all the auxiliary fields necessary for a gauge invariant description is the moral equivalent of the ghost coordinate in the BRST formalism although the precise connection is not clear. The origin of the critical dimension and mass spectrum are not clear. These are free parameters and can be chosen to match string theory. It is possible that quantum consistency will force this on us. But at the classical level there do not seem to be any such constraints.

The Q-rules need to be extended to higher levels. The present method is very tedious and leads to a highly over determined set of linear equations that miraculously seem to have a solution. This suggests that some underlying pattern lies unnoticed. Furthermore the fact that the rules can be made consistent with dimensional reduction is also not *a priori* clear and again points to some underlying structure.

Finally and perhaps most importantly the gauge transformations have a simple form of local (along the string) scale transformations of the generalized loop variable momenta. This was in fact the original motivation for this formalism. The possible geometrical space time interpretation in terms of a *space time* renormalization group (with a finite cutoff) as the symmetry group of string theory is described in [23]. The speculation described there is that string theory should be thought of as a method of regularizing a field theory by a Lorentz invariant cutoff and therefore should be equivalent to a dynamical space time lattice.

The gauge transformations for the open string have an Abelian form which means the interactions do not modify it. The interactions are in terms of gauge invariant field strengths - generalizations of Maxwell field strengths. This is in contrast to BRST string field theory [32]. For closed strings this continues to be the case for the massive fields. But for the massless field this field strength is not gauge invariant unless the gauge transformation is modified to a non Abelian form. This then becomes general coordinate transformations.

One can speculate on the possible space time interpretation of the other gauge symmetries of the closed string. For instance in Sec 7.2.1 and [33] it was noted that the gauge parameters for the graviton and antisymmetric tensor seem to have the natural interpretation of the real and imaginary parts of a complex parameter. This leads naturally to a speculation that space time coordinates could be complex. Such a speculation has been made earlier also [50, 51]

The massive modes have higher gauge symmetries. These are realized in a "broken" form with Stuckelberg fields. However we should remember that a massive spin 2 field with a Stuckelberg realization is what is naturally described by the formalism. To make the graviton massless we had to extend the gauge transformations to a "non Abelian" form. It is possible that extending the higher gauge symmetries to a non Abelian form will allow the higher spin modes to become massless as well. This would then describe a different theory - or a different phase of the same theory - perhaps a more symmetric form.

We hope to return to these questions soon.

Appendix A Riemann Normal Coordinates and Tensors

We review some facts about Riemann Normal Coordinates [44, 45, 46].

A.1 Defining the Coordinate System

1. In a manifold (with metric), first pick a coordinate system. Call it x^i and let the metric be g_{ij} (We also occasionally use a or μ for the indices). Pick an origin O at a point labelled as x_0. Draw geodesics through O. The geodesics are labelled by their unit tangent vectors at O, ξ^i and parametrized by s the proper distance along the geodesic. The equation for the geodesic is

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0 \tag{A.1.1}$$

One can also write a Taylor expansion:

$$x^a(s) = x^a_0 + s\frac{dx^a(s)}{ds}|_{s=0} + \frac{s^2}{2!}\frac{d^2x^a(s)}{ds^2}|_{s=0} + ... \tag{A.1.2}$$

Clearly $\frac{dx^a(s)}{ds}|_{s=0} = \xi^a$. Similarly

$$\frac{d^2x^a(s)}{ds^2}|_{s=0} = \Gamma^a_{bc}(x_0)\dot{x}^b(0)\dot{x}^c(0) = \Gamma^a_{bc}(x_0)\xi^b\xi^c$$

By differentiating (A.1.1) we get

$$\begin{aligned}\frac{d^3x^a(s)}{ds^3} &= -\frac{d}{ds}[\Gamma^a_{bc}\dot{x}^b\dot{x}^c] = -(\partial_d\Gamma^a_{bc})\dot{x}^b\dot{x}^c\dot{x}^d - \Gamma^a_{bc}\frac{d}{ds}[\dot{x}^b\dot{x}^c] \\ &= -\underbrace{[\partial_d\Gamma^a_{bc} - \Gamma^a_{bi}\Gamma^i_{dc} - \Gamma^a_{ic}\Gamma^i_{db}]}_{\Gamma^a_{bcd}(s)}\dot{x}^b\dot{x}^c\dot{x}^d \\ &= -\frac{1}{3!}\Gamma^a_{(bcd)}(s)\dot{x}^b\dot{x}^c\dot{x}^d \equiv \tilde{\Gamma}^a_{bcd}(s)\dot{x}^b\dot{x}^c\dot{x}^d\end{aligned} \tag{A.1.3}$$

The geodesic equation (A.1.1) has been used and we have symmetrized on the indices in the last step. A similar calculation for $\frac{d^4x^a(s)}{ds^4}$ gives

$$\begin{aligned}\frac{d^4x^a(s)}{ds^4} &= -\frac{d}{ds}[\tilde{\Gamma}^a_{bcd}\dot{x}^b\dot{x}^c\dot{x}^d] = -(\partial_e\tilde{\Gamma}^a_{bcd})\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e - \tilde{\Gamma}^a_{bcd}\frac{d}{ds}[\dot{x}^b\dot{x}^c\dot{x}^d] \\ &= -\underbrace{[\partial_e\tilde{\Gamma}^a_{bcd} - \tilde{\Gamma}^a_{bid}\Gamma^i_{ec} - \tilde{\Gamma}^a_{icd}\Gamma^i_{eb} - \tilde{\Gamma}^a_{bci}\Gamma^i_{ed}]}_{\Gamma^a_{bcde}(s)}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e \\ &= -\frac{1}{4!}\Gamma^a_{(bcde)}(s)\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e \equiv \tilde{\Gamma}^a_{bcde}(s)\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\end{aligned} \tag{A.1.4}$$

Plug these into (A.1.2) to get

$$x^a(s) = x^a_0 + s\xi^a + \frac{s^2}{2!}\Gamma^a_{bc}(0)\xi^b\xi^c + \frac{s^3}{3!}\tilde{\Gamma}^a_{bcd}(0)\xi^b\xi^c\xi^d + \frac{s^4}{4!}\tilde{\Gamma}^a_{bcde}(0)\xi^b\xi^c\xi^d\xi^e + ... \tag{A.1.5}$$

It can also be shown that

$$\underbrace{\Gamma^a_{bcd}}_{\square\otimes\square\square} = \underbrace{\tilde{\Gamma}^a_{bcd}}_{\square\square\square} + \frac{1}{3}\underbrace{(R^a{}_{bdc} + R^a{}_{dbc})}_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}} \tag{A.1.6}$$

Since the Riemann tensor has some antisymmetric indices, it drops out of the Taylor expansion (A.1.5).

2. Consider an arbitrary point,P, labelled x in the original coordinate system. We give it a new label: Consider the geodesic starting from the origin,O, and going through P. Let the unit tangent vector be ξ^i. Let $y^i = s\xi^i$. Choose y^i to label this point. The y defines a coordinate system called Riemann Normal Coordinates (RNC). The relation between x and y is obtained by rewriting (A.1.5) in terms of y.
$$x^a(y) = x_0^a + y^a + \frac{1}{2!}\Gamma^a_{bc}(0)y^by^c + \frac{1}{3!}\tilde{\Gamma}^a_{bcd}(0)y^by^cy^d + \frac{1}{4!}\tilde{\Gamma}^a_{bcde}(0)y^by^cy^dy^e + ... \tag{A.1.7}$$

A.2 Properties of Γ

Let us put bars to denote quantities in the RNC. (For eg. $\bar{\Gamma}^a_{bc}$). Then if we write (A.1.5) for geodesics in the RNC y, then only the first term survives : $y^a = s\xi^a$. Thus all the $\bar{\tilde{\Gamma}}^a_{bc..}(0)\xi^b\xi^c... = 0$ at the origin of the coordinate system. But at the origin ξ^a can point in any direction. Therefore it must be true that *at the origin*:
$$\bar{\Gamma}^a_{bc}(0) = \bar{\tilde{\Gamma}}^a_{bcd}(0) = \bar{\tilde{\Gamma}}^a_{bcde}(0) = ... = 0 \tag{A.2.8}$$

Furthermore, all along the geodesic, in the RNC, $\frac{d^ny^a}{ds^n} = 0$ - because the geodesics through the origin are straight lines. Thus from the geodesic equation (A.1.1) and its derivatives we can conclude that *all along the geodesic* specified by ξ^a:
$$\bar{\tilde{\Gamma}}^a_{bcde...}(y)\xi^b\xi^c\xi^d\xi^e... = 0 \tag{A.2.9}$$

That (A.2.8) is a complete specification of the freedom of coordinate transformation is obvious from counting parameters in the coordinate transformation $x'(x)$. If we write this as a Taylor series
$$x'^a = \frac{\partial x'^a}{\partial x^b}|_{x=0}x^b + \frac{1}{2!}\frac{\partial^2 x'^a}{\partial x^b\partial x^c}|_{x=0}x^bx^c + \frac{1}{3!}\frac{\partial^3 x'^a}{\partial x^b\partial x^c\partial x^d}|_{x=0}x^bx^cx^d + ...+$$
At each order the coefficients of the Taylor expansion have a tensor structure with one upper index a, followed by completely symmetrized lower indices $b, c, d....$ This is exactly the index structure of the $\tilde{\Gamma}^a_{bcd...}$. Thus their numbers are exactly the same in any dimension.

A.3 Expansion of Tensors

A.3.1 Scalar

Consider first the Taylor expansion of a scalar field about a point O labelled by x_0. Let $\Delta x^\mu = x^\mu - x_0^\mu$. Then
$$\phi(x) = \phi(x_0) + \Delta x^\mu\partial_\mu\phi(x_0) + \frac{1}{2!}\Delta x^\mu\Delta x^\nu\partial_\mu\partial_\nu\phi(x_0) + ... \tag{A.3.10}$$

Let us rewrite ordinary derivatives in terms of covariant derivatives:
$$\begin{aligned}\partial_\mu\phi &= \nabla_\mu\phi \\ \partial_\mu\partial_\nu\phi &= \nabla_\mu(\nabla_\nu\phi) + \Gamma^\rho_{\mu\nu}\nabla_\rho\phi\end{aligned} \tag{A.3.11}$$

Similarly after some algebra
$$\partial_\rho\partial_\mu\partial_\nu\phi = \nabla_\rho\nabla_\mu\nabla_\nu\phi + \Gamma^\lambda_{(\rho\mu}\nabla_{|\lambda|}\nabla_{\nu)}\phi + [\tilde{\Gamma}^\lambda_{\mu\nu\rho} + \Gamma^\lambda_{(\mu|\sigma|}\Gamma^\sigma_{\rho\nu)}]\nabla_\lambda\phi \tag{A.3.12}$$

In a general coordinate system Δx^μ is not a tensor, hence this is not a covariant expansion in terms of tensors - as the explicit presence of Γ's shows.

The solution is well known: work with $y^\mu = s\xi^\mu$. In this case the relation between x and y is as given above:
$$x^a(y) = x_0^a + y^a + \frac{1}{2!}\Gamma^a_{bc}(0)y^by^c + \frac{1}{3!}\tilde{\Gamma}^a_{bcd}(0)y^by^cy^d + \frac{1}{4!}\tilde{\Gamma}^a_{bcde}(0)y^by^cy^dy^e + ... \tag{A.3.13}$$

We use bars over quantities to indicate that they are written in a RNC. Scalars do not transform so we do not need a bar. In the RNC, all the $\bar{\Gamma}$'s in (A.3.12) vanish at the origin so one obtains a covariant expansion, because the y^μ are geometric objects - vectors - at the origin:

$$\phi(y) = \phi(0) + y^\mu \bar{\nabla}_\mu \phi(0) + \frac{1}{2!} y^\mu y^\nu \bar{\nabla}_\mu \bar{\nabla}_\nu \phi(0) + \frac{1}{3!} y^\mu y^\nu y^\rho \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\nabla}_\rho \phi(0)...$$

$$= \phi(0) + y^\mu \partial_\mu \phi(0) + \frac{1}{2!} y^\mu y^\nu \partial_\mu \partial_\nu \phi(0) + \frac{1}{3!} y^\mu y^\nu y^\rho \partial_\mu \partial_\nu \partial_\rho \phi(0)... \tag{A.3.14}$$

i.e. in the RNC, for a scalar, the ordinary Taylor expansion is an expansion in covariant derivatives. The LHS is a scalar at y. Each term on the RHS is a scalar *at the origin* $y = 0$. Nevertheless, since both sides are scalars, one can transform the first equation, which is manifestly covariant, to any other coordinate system, x, with the understanding that y^μ transforms as a vector into:

$$\delta x^\nu = \frac{\partial x^\nu}{\partial y^\mu}|_{y=0} y^\mu$$

This $\delta x^\nu \neq \Delta x^\mu (\equiv x^\mu - x_0^\mu)$.

A.3.2 Vector

One can perform the same set of steps for a vector:

$$S_\mu(x_0 + \Delta x) = S_\mu(x_0) + \Delta x^\rho \partial_\rho S_\mu(x_0) + \frac{1}{2} \Delta x^\rho \Delta x^\sigma \partial_\rho \partial_\sigma S_\mu(x_0) + ...$$

$$= S_\mu(x_0) + \Delta x^\rho [\nabla_\rho S_\mu(x_0) + \Gamma^\nu_{\rho\mu} S_\nu(x_0)] + \frac{1}{2} \Delta x^\rho \Delta x^\sigma [\nabla_\rho \nabla_\sigma S_\mu(x_0) + \Gamma^\lambda_{\sigma\rho} \nabla_\lambda S_\mu(x_0) + 2\Gamma^\lambda_{\rho\mu} \nabla_\sigma S_\lambda(x_0)]$$

$$+ \frac{1}{2} \Delta x^\rho \Delta x^\sigma [\tilde{\Gamma}^\nu_{\sigma\mu\rho}(x_0) + \frac{1}{3} R^\nu{}_{\sigma\rho\mu}(x_0) + \Gamma^\nu_{(\sigma|\lambda|} \Gamma^\lambda_{\mu\rho)}(x_0)] S_\nu(x_0) + ... \tag{A.3.15}$$

The equation is manifestly not covariant because of the presence of the explicit Γ's. Once again we can rewrite this equation in the RNC, y. All the $\bar{\Gamma}$'s are zero (at the origin, $y = 0$) and we get

$$\bar{S}_\mu(y) = \bar{S}_\mu(0) + y^\rho \bar{\nabla}_\rho \bar{S}_\mu(0) + \frac{1}{2} y^\rho y^\sigma [\bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{S}_\mu(0) + \frac{1}{3} \bar{R}^\nu{}_{\sigma\rho\mu}(0) \bar{S}_\nu(0)] + ... \tag{A.3.16}$$

The main difference with the scalar is the appearance of the Riemann tensor and its derivatives.

Note that the LHS is a vector at the general point y whereas the RHS is a sum of vectors at the *origin* $y = 0$. Thus this equation is valid only in the RNC. If we want transform to a different coordinate system the LHS and RHS transform differently, because they are vectors at *different* points.

We can write for the LHS

$$\bar{S}_\nu(y) = \frac{\partial x^\mu}{\partial y_\nu}|_y S_\mu(x)$$

and for the RHS we can write similarly

$$\bar{S}_\nu(0) = \frac{\partial x^\mu}{\partial y_\nu}|_0 S_\mu(x_0)$$

Plugging in these transformation matrices, one can relate $S_\mu(x)$ to $S_\mu(x_0)$ and its derivatives.

A.3.3 Tensor

$$\bar{W}_{\alpha_1....\alpha_p}(y) = \bar{W}_{\alpha_1....\alpha_p}(0) \;+\; \bar{W}_{\alpha_1....\alpha_p,\mu}(0)y^\mu \;+$$
$$\frac{1}{2!}\{\bar{W}_{\alpha_1....\alpha_p,\mu\nu}(0) \;-\; \frac{1}{3}\sum_{k=1}^{p}\bar{R}^{\beta}{}_{\mu\alpha_k\nu}(0)\bar{W}_{\alpha_1..\alpha_{k-1}\beta\alpha_{k+1}..\alpha_p}(0)\}y^\mu y^\nu \;+$$
$$\frac{1}{3!}\{\bar{W}_{\alpha_1....\alpha_p,\mu\nu\rho}(0) - \sum_{k=1}^{p}\bar{R}^{\beta}{}_{\mu\alpha_k\nu}(0)\bar{W}_{\alpha_1..\alpha_{k-1}\beta\alpha_{k+1}..\alpha_p,\rho}(0)$$
$$-\frac{1}{2}\sum_{k=1}^{p}\bar{R}^{\beta}{}_{\mu\alpha_k\nu,\rho}(0)\bar{W}_{\alpha_1..\alpha_{k-1}\beta\alpha_{k+1}..\alpha_p}(0)\}y^\mu y^\nu y^\rho + ... \tag{A.3.17}$$

The commas denote covariant derivatives. The bars remind us that we are in an RNC.

A.4 Expansion of Vertex Operators

We have seen that it is easier to work with scalars. In the world sheet action the terms are all scalars, because the tensor indices are contracted with those of the vertex operators. For eg $S_\mu(X(z,))\partial_z X^\mu(z)$. We use $X(z), Y(z)$ to denote the world sheet fields instead of x and $\bar{Y}$ instead of RNC y. So the combined object is a scalar (in space time). So we study the expansion of vertex operators. This is required when one performs an OPE of vertex operators in the interaction term of the ERG. It is important to do this covariantly.

The value of X is parametrized by z (and $\bar{z}$ for closed strings). So a Taylor expansion in z bcomes a Taylor expansion in X. Since z is a space time scalar we are guaranteed that the expansion coefficients will be scalars.In loop variables, we have $Y(z,x_n)$ and the Taylor expansion is in x_n. We denote these variables generically by z^α. Thus let us choose the origin, x_0, of the RNC system such that $X(0)=x_0$. Then

$$X^i(z) - X^i(0) = \Delta X^i = z^\alpha\partial_\alpha X^i(0) + \frac{z^\alpha z^\beta}{2!}\partial_\alpha\partial_\beta X^i(0) + \frac{z^\alpha z^\beta z^\gamma}{3!}\partial_\alpha\partial_\beta\partial_\gamma X^i(0) + ... \tag{A.4.18}$$

Now $X^i_\alpha \equiv \partial_\alpha X^i$ is a vector:

$$\frac{\partial X'^i(z)}{\partial z^\alpha} = \frac{\partial X'^i(z)}{\partial X^j(z)}\frac{\partial X^j(z)}{\partial z^\alpha}$$

However $\frac{\partial^2 X^i}{\partial z^\alpha \partial z^\beta}$ is not a tensor. Define a covariant derivative:

$$D_\beta X^i_\alpha(z) \equiv \partial_\beta X^i_\alpha(z) + \Gamma^i_{ab}(X(z))X^a_\beta(z)X^b_\alpha(z) \tag{A.4.19}$$

(We will for convenience use $a,b,c..$ for the dummy indices and $i,j,k..$ for the uncontracted indices.) Then

$$\partial_\beta X^i_\alpha(z) = D_\beta X^i_\alpha(z) - \Gamma^i_{ba}(X(z))X^b_\beta(0)X^a_\alpha(0) \tag{A.4.20}$$

Clearly in RNC, at the origin, $\bar{\Gamma}^i_{ba}(X(0)) = 0$,

$$\partial_\beta \bar{X}^a_\alpha(0) = \bar{D}_\beta \bar{X}^a_\alpha(0) \tag{A.4.21}$$

$$\begin{aligned}
\partial_\gamma\partial_\beta X^i_\alpha(z) &= D_\gamma D_\beta X^i_\alpha - \Gamma^i_{ca}X^c_\gamma D_\beta X^a_\alpha - \partial_\gamma[\Gamma^i_{ba}X^b_\beta X^a_\alpha] \\
&= D_\gamma D_\beta X^i_\alpha - \Gamma^i_{ca}X^c_\gamma D_\beta X^a_\alpha - (\partial_\gamma\Gamma^i_{ba})X^b_\beta X^a_\alpha - \Gamma^i_{ba}\partial_\gamma[X^b_\beta X^a_\alpha] \\
&= D_\gamma D_\beta X^i_\alpha - \Gamma^i_{ca}X^c_\gamma D_\beta X^a_\alpha - (\partial_\gamma\Gamma^i_{ba})X^b_\beta X^a_\alpha - \Gamma^i_{ba}D_\gamma[X^b_\beta X^a_\alpha] - \Gamma^i_{ba}[-\Gamma^b_{cd}X^c_\gamma X^d_\beta X^a_\alpha - \Gamma^a_{cd}X^c_\gamma X^b_\beta X^d_\alpha] \\
&= D_\gamma D_\beta X^i_\alpha - \Gamma^i_{ca}X^c_\gamma D_\beta X^a_\alpha - \Gamma^i_{ba}D_\gamma[X^b_\beta X^a_\alpha] - (\partial_c\Gamma^i_{ba})X^c_\gamma X^b_\beta X^a_\alpha - \Gamma^i_{ba}[-\Gamma^b_{cd}X^c_\gamma X^d_\beta X^a_\alpha - \Gamma^a_{cd}X^c_\gamma X^b_\beta X^d_\alpha] \\
&= D_\gamma D_\beta X^i_\alpha(z) - \Gamma^i_{ca}(X(z))X^c_\gamma D_\beta X^a_\alpha(z) - \Gamma^i_{ba}(X(z))D_\gamma[X^b_\beta X^a_\alpha(z)] - \Gamma^i_{bac}(X(z))X^c_\gamma X^b_\beta X^a_\alpha(z)
\end{aligned}$$

We can now symmetrize the RHS in α, β, γ because the LHS is symmetric, and write

$$\partial_\gamma \partial_\beta X^i_\alpha = \frac{1}{3!}\{D_{(\gamma} D_\beta X^i_{\alpha)} - \Gamma^i_{ca} X^c_{(\gamma} D_\beta X^a_{\alpha)} - \Gamma^i_{ba} D_{(\gamma}[X^b_\beta X^a_{\alpha)}] - \tilde{\Gamma}^i_{bac} X^c_{(\gamma} X^b_\beta X^a_{\alpha)}\} \tag{A.4.22}$$

For the last term we have used the fact that $X^c_{(\gamma} X^b_\beta X^a_{\alpha)}$ is also symmetric now in a, b, c. It is clear from the above pattern that symmetrized vertex operators will involve the $\tilde{\Gamma}^a_{bcde...}$ as defined in the last subsection.

If we now specialize to RNC, where $\bar{\tilde{\Gamma}}^a_{bcde...} = 0$, we find that, again at the origin,

$$\partial_\gamma \partial_\beta \bar{X}^i_\alpha(0) = \frac{1}{3!} \bar{D}_{(\gamma} \bar{D}_\beta \bar{X}^i_{\alpha)}(0) \tag{A.4.23}$$

Let us do one more:

$$\begin{aligned}
\partial_\delta \partial_\gamma \partial_\beta X^i_\alpha \quad &= \quad \frac{1}{4!}\{D_{(\delta} D_\gamma D_\beta X^i_{\alpha)} - \Gamma^i_{da} X^d_{(\delta} D_\gamma D_\beta X^a_{\alpha)}\} \\
&\quad - \frac{1}{8}\{[\tilde{\Gamma}^i_{cad} - \frac{1}{6}(R^i{}_{dac} + R^i{}_{cad})] X^d_{(\delta} X^c_\gamma D_\beta X^a_{\alpha)} + \Gamma^i_{ca} D_{(\delta} X^c_\gamma D_\beta X^a_{\alpha)}\} \\
&\quad - \frac{1}{4!}\{\tilde{\Gamma}^i_{abcd} X^d_{(\delta} X^c_\gamma X^b_\beta X^a_{\alpha)} + \tilde{\Gamma}^i_{abc} D_{(\gamma}(X^c_\gamma X^b_\beta X^a_{\alpha)})\}
\end{aligned} \tag{A.4.24}$$

Note that we have used (A.1.6) to write Γ^i_{cad} in terms of $\tilde{\Gamma}^i_{cad}$ and the Riemann tensor. Once again specializing to RNC, at the origin, we get

$$\partial_\delta \partial_\gamma \partial_\beta \bar{X}^i_\alpha(0) = \frac{1}{4!} \bar{D}_{(\delta} \bar{D}_\gamma \bar{D}_\beta \bar{X}^i_{\alpha)}(0) + \frac{1}{48}(\bar{R}^i{}_{dac}(0) + \bar{R}^i{}_{cad}(0))] X^d_{(\delta} \bar{X}^c_\gamma \bar{D}_\beta \bar{X}^a_{\alpha)}(0)$$

Thus the Taylor expansion in an RNC is

$$\bar{X}^i(z) = z^\alpha \bar{X}^i_\alpha(0) + \frac{z^\alpha z^\beta}{2!} D_\beta \bar{X}^a_\alpha(0) + \frac{z^\alpha z^\beta z^\gamma}{3!} \frac{1}{3!} D_{(\gamma} D_\beta \bar{X}^i_{\alpha)}(0)$$

$$+ \frac{z^\alpha z^\beta z^\gamma z^\delta}{4!}\{\frac{1}{4!} D_{(\delta} D_\gamma D_\beta \bar{X}^i_{\alpha)}(0) + \frac{1}{48}(\bar{R}^i{}_{dac}(0) + \bar{R}^i{}_{cad}(0))] \bar{X}^d_{(\delta} \bar{X}^c_\gamma D_\beta \bar{X}^a_{\alpha)}(0)\} \tag{A.4.25}$$

A.5 Expansion of Scalar Combination of Tensor and Vertex Operator

A.5.1 Scalar

The simplest case is again a scalar $\phi(X(z))$ which we can expand in powers of z^α. We start with

$$\phi(x) = \phi(x_0) + \Delta x^\mu \partial_\mu \phi(x_0) + \frac{1}{2!} \Delta x^\mu \Delta x^\nu \partial_\mu \partial_\nu \phi(x_0) + ... \tag{A.5.26}$$

and substitute for the ordinary derivatives expressions such as (A.3.12) :

$$\phi(X(z)) = \phi(X(0)) + \Delta X^\mu \nabla_\mu \phi(X(0)) + \frac{1}{2!} \Delta X^\mu \Delta X^\nu (\nabla_\mu \nabla_\nu \phi(X(0)) + \Gamma^\rho_{\mu\nu}(X(0)) \nabla_\rho \phi(X(0))$$

$$+ \frac{1}{3!} \Delta X^\mu \Delta X^\nu \Delta X^\rho \{\nabla_\rho \nabla_\mu \nabla_\nu \phi(X(0)) + \Gamma^\lambda_{(\rho\mu}(X(0)) \nabla_{|\lambda|} \nabla_{\nu)} \phi(X(0)) + [\tilde{\Gamma}^\lambda_{\mu\nu\rho} + \Gamma^\lambda_{(\mu|\sigma|} \Gamma^\sigma_{\rho\nu)}(X(0))] \nabla_\lambda \phi(X(0))\} + ... \tag{A.5.27}$$

This equation does not look covariant because ΔX^μ is not a covariant object. For ΔX^μ we substitute (A.4.18) which is a sum of non covariant terms multiplied by powers of z^α. But the final expression has a scalar on the LHS and the RHS is an expansion in z^α which is a space time scalar. Accordingly each

term in the sum must be a scalar. We in fact do find that all the explicit non covariant Γ's cancel amongst themselves and the result is manifestly covariant. We give the first few terms:

$$
\begin{aligned}
\phi(X(z)) &= \phi(X(0)) + z^\alpha X^i_\alpha \nabla_i \phi(X(0)) + \frac{z^\alpha z^\beta}{2!}(D_\beta X^i_\alpha \nabla_i \phi(X(0)) + X^i_\alpha X^j_\beta \nabla_i \nabla_j \phi(X(0))) \\
&+ z^\alpha z^\beta z^\gamma \{[\frac{1}{3!}\frac{1}{6} D_{(\alpha} D_\beta X^i_{\gamma)}] \nabla_i \phi(X(0)) + \frac{1}{4} X^{(i}_\alpha D_\beta X^{j)}_\gamma \nabla_i \nabla_j \phi(X(0)) \\
&+ \frac{1}{3!} X^i_\alpha X^j_\beta X^k_\gamma \nabla_i \nabla_j \nabla_k \phi(X(0))\} \\
&+ ....
\end{aligned}
\tag{A.5.28}
$$

This calculation has been done in a general coordinate system. Nevertheless, it could have been done in an RNC, where neither the expansion for ϕ, nor the expansion for ΔX has any Γ's - both expansions look manifestly covariant. If we combine these expansions, we must necessarily obtain the same equation. Thus

$$
\phi(\bar{Y}(z)) = \phi(0) + \bar{Y}^i(z)\bar{\nabla}_i\phi(0) + \frac{1}{2!}\bar{Y}^i(z)\bar{Y}^j(z)\bar{\nabla}_i\bar{\nabla}_j\phi(0) + \frac{1}{3!}\bar{Y}^i(z)\bar{Y}^j(z)\bar{Y}^k(z)\bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_k\phi(0)... \tag{A.5.29}
$$

and

$$
\bar{Y}^i(z) = z^\alpha \bar{Y}^i_\alpha(0) + \frac{z^\alpha z^\beta}{2!}\bar{D}_\beta \bar{Y}^a_\alpha(0) + \frac{z^\alpha z^\beta z^\gamma}{3!}\frac{1}{3!}\bar{D}_{(\gamma}\bar{D}_\beta \bar{Y}^i_{\alpha)}(0)
$$

$$
+ \frac{z^\alpha z^\beta z^\gamma z^\delta}{4!}\{\frac{1}{4!}\bar{D}_{(\delta}\bar{D}_\gamma \bar{D}_\beta Y^i_{\alpha)}(0) + \frac{1}{48}(\bar{R}^i{}_{dac}(0) + \bar{R}^i{}_{cad}(0))]\bar{Y}^d_{(\delta}\bar{Y}^c_\gamma \bar{D}_\beta \bar{Y}^a_{\alpha)}(0)\} \tag{A.5.30}
$$

As explained above, while these equations look covariant, they are in fact valid only in RNC. Nevertheless it is easy to check that when (A.5.29) and (A.5.30) are combined we do get the same expansion (B.1.8) but with bars - i.e. in the RNC. But now this expression is a sum of *scalars* on the RHS and a *scalar* on the LHS (albeit at different points) so we can remove the bars and use this equation in any coordinate system.

A.5.2 Vector

Let us now consider the vector: $S_i(X(z))\partial_\alpha X^i(z) = S_i(X(z))X^i_\alpha(z)$. We use (A.3.15) for the expansion of the vector in powers of ΔX^i and (A.4.18) for ΔX^i. Both have non covariant terms but the non covariant terms cancel amongst themselves and the result is a manifestly covariant expansion in powers of z^α.

$$
\begin{aligned}
S_i(X(z))X^i_\alpha(z) &= S_i(X(0))X^i_\alpha(0) + z^\beta[\frac{1}{2}S_i(X(0))D_{(\beta}X^i_{\alpha)}(0) + \nabla_j S_i(X(0))X^i_\alpha X^j_\beta(0)] \\
&\frac{z^\beta z^\alpha}{2!}[\frac{1}{2}\nabla_j S_i(X(0))D_{(\beta}X^i_{\alpha)} + (\frac{\nabla_{(i}\nabla_{j)}S_i(X(0))}{2} + \frac{1}{3}R^l{}_{jki}(X(0))S_l(X(0)))X^k_\beta X^j_\gamma X^i_\alpha(0) + \\
&+ \frac{D_{(\alpha}D_\beta X^i_{\gamma)}(0)S_i(X(0))}{6} + (\nabla_j S_i(X(0)))D_{(\beta}X_{\alpha)})^i X^j_\gamma(0)] \\
&+ ...
\end{aligned}
\tag{A.5.31}
$$

As with the scalar, if one works in RNC, one gets the manifestly covariant expression directly, without having to worry about cancellations amongst the non covariant terms.

As mentioned above, these expansions are required when one performs OPE in the interaction term of the ERG.

A.6 Covariantizing $\bar{Y}^\mu(z)$

$\bar{Y}^\mu(z)$ is the RNC space time coordinate field that occurs in the world sheet theory. As explained above at some length it is a geometric object defined at the origin of the RNC and is tangent to the geodesic that starts from the origin, O, of the RNC and goes through a given point, P, with RNC coordinate $\bar{Y}^\mu$. It is thus a coordinate as well as a vector at the origin O. One may well ask if there is a geometric object at P, that coincides with this in the RNC. Note that $\bar{Y}^\mu$ is a tangent to the geodesic at P as well, because in the RNC the geodesics are straight lines. Thus one can also think of $\bar{Y}^\mu$ as a geometric object at P - viz the tangent vector to the geodesic at P, and transform it as a vector, into any other coordinate system. This is useful because one needs a covariant definition of the Green function, that in the RNC is $\langle \bar{Y}^\mu(z)\bar{Y}^\nu(w)\rangle$.

We elaborate on this idea: As before let X be a general coordinate system. At a point O (with coordinates X_0) we set the origin of an RNC system $\bar{Y}^\mu$. The point O has coordinate $\bar{Y}^\mu = 0$. For a general point P with coordinate X, we consider a geodesic that starts from O and goes through P. Let the tangent vector to this geodesic at O be $\vec{\xi_P}$ and the proper distance along this geodesic to P be t_P. Then $\bar{Y}^\mu = t_P\xi^\mu_P$. $\vec{\xi_P}$ is a geometric object - a vector at O, *not* at P. So $\bar{Y}^\mu$ transforms as a vector at O. One would like an object that is a vector at P. So let us define the tangent vector field, $\xi^\mu(P)$ (or $\xi^\mu(X_P)$) of unit norm vectors tangent to the geodesics through O at the (general) point P. They obey

$$\xi^\nu\nabla_\nu\xi^\mu = \xi^\nu\frac{\partial\xi^\mu}{\partial X^\nu} + \Gamma^\nu_{\mu\rho}\xi^\mu\xi^\rho = 0$$

In the RNC this equation becomes

$$\bar{\xi}^\nu\nabla_\nu\bar{\xi}^\mu = \bar{\xi}^\nu\frac{\partial\bar{\xi}^\mu}{\partial X^\nu} + \bar{\Gamma}^\nu_{\mu\rho}\bar{\xi}^\mu\bar{\xi}^\rho = 0$$

But we know that in the RNC all along the geodesic,

$$\bar{\Gamma}^\nu_{\mu\rho}\bar{\xi}^\mu\bar{\xi}^\rho = 0$$

This follows from the fact that $\bar{Y}^\mu$ satisfies the geodesic equation

$$\frac{d^2\bar{Y}^\mu}{dt^2} + \bar{\Gamma}^\nu_{\mu\rho}\frac{d\bar{Y}^\mu}{dt}\frac{d\bar{Y}^\rho}{dt} = 0$$

and since $\frac{d^2\bar{Y}^\mu}{dt^2} = 0$ we get $\bar{\Gamma}^\nu_{\mu\rho}\frac{d\bar{Y}^\mu}{dt}\frac{d\bar{Y}^\rho}{dt} = \bar{\Gamma}^\nu_{\mu\rho}\bar{\xi}^\mu\bar{\xi}^\rho = 0$.

Thus we get

$$\bar{\xi}^\nu\frac{\partial\bar{\xi}^\mu}{\partial X^\nu} = 0$$

which means $\bar{\xi}^\mu$ is constant along a geodesic. A solution to this is thus the constant (along a geodesic) vector field $\bar{\xi}^\mu(\bar{Y}_P) = \bar{\xi}_P$. This is just the obvious fact that in the RNC geodesics are straight lines through the origin, so the tangent vector field is a constant (along a geodesic) vector field. We thus see that in the RNC $\bar{Y}^\mu_P = t_P\bar{\xi}^\mu(\bar{Y}_P)$ is not only a coordinate, it is also a vector field, i.e. the two objects coincide. This will not be the case in a general coordinate system.

Thus when we change coordinates to Y, the vector field $\bar{Y}^\mu(\bar{Y}_P) = t_P\bar{\xi}^\mu(\bar{Y}_P)$ transforms like a vector field *at P* to a new vector field,

$$y^\mu(P) \equiv t_P\xi^\mu(Y_P) = t_P\frac{\partial Y^\mu}{\partial\bar{Y}^\nu}|_P\bar{\xi}^\nu(P) \tag{A.6.32}$$

whereas the coordinate $\bar{Y}^\mu_P$ becomes Y^μ_P.

Thus when we see an expression involving $\bar{Y}^\mu = t_P\bar{\xi}^\mu$ in the RNC, there are two distinct geometric objects that it can correspond to in a general coordinate system: $y^\mu(O) \equiv t_P\xi^\mu(O)$ or $y^\mu(P) \equiv t_p\xi^\mu(P)$. (This is in addition to the original interpretation as a *coordinate*, in which case it just becomes X^μ in a new coordinate system.)

Appendix B Free Equation

B.1

The details of the calculation of the Level 2 (graviton) and Level 4 free equation are given here.

We have to evaluate the second derivative, which is given by the action of a functional derivative on (5.4.1):

$$
\begin{gathered}
= \int dz' \int dz'' \; \dot{G}(z',z'') \\
\eta^{\mu\nu}\int du \; [\frac{\partial}{\partial Y^\nu(u)}\delta(u-z') + [\frac{\partial}{\partial x_1}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{1;0}(u)} + \\
[\frac{\partial}{\partial \bar{x}_1}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{0;\bar{1}}(u)} + [\frac{\partial}{\partial x_2}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{2;0}(u)} + \\
[\frac{\partial}{\partial \bar{x}_2}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{0;\bar{2}}(u)} + [\frac{\partial^2}{\partial x_1 \partial \bar{x}_1}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{1;\bar{1}}(u)} + ...] \\
\Big\{ \underbrace{\frac{\partial \mathcal{L}[Y(u), Y_{n,\bar{m}}(u)]}{\partial Y^\mu(u)}\delta(u-z'')}_{I} - \underbrace{\partial_{x_1}\frac{\partial \mathcal{L}[Y(u), Y_{n;\bar{m}}(u)]}{\partial Y^\mu_{1;0}(u)}\delta(u-z'')}_{II} \\
\underbrace{-\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u), Y_{n;\bar{m}}(u)]}{\partial Y^\mu_{0;\bar{1}}(u)}\delta(u-z'')}_{III} + \underbrace{\partial_{x_1}\partial_{\bar{x}_1}\frac{\partial \mathcal{L}[Y(u), Y_{n,\bar{m}}(u)]}{\partial Y^\mu_{1,\bar{1}}(u)}\delta(u-z'')}_{IV} \Big\}
\end{gathered}
\tag{B.1.1}
$$

Let us evaluate the action of the derivatives on each of the four terms labeled I,II,III and IV. The result has to be symmetric in $z' \leftrightarrow z''$ and also for every term, there is also a corresponding complex conjugate term. This fact reduces the number of independent terms to be evaluated. (Our notation is: x_n refers to u, x'_n refers to z' and x''_n refers to z''. Thus for instance, $\frac{\partial \delta(u-z')}{\partial x_n} = -\frac{\partial \delta(u-z')}{\partial x'_n}$)

1.
$$
\int du \; \eta^{\mu\nu}\frac{\partial}{\partial Y^\nu(u)}\frac{\partial \mathcal{L}[Y(u), Y_{n,\bar{m}}(u)]}{\partial Y^\mu(u)}\delta(u-z')\delta(u-z'')
$$
$$
= -k_0^2\mathcal{L}(z')\delta(z'-z'') \tag{B.1.2}
$$

2.
$$
\begin{aligned}
&\int du \; \eta^{\mu\nu}\Big([\frac{\partial}{\partial x_1}\delta(u-z')]\frac{\partial}{\partial Y^\nu_{1;0}(u)}\frac{\partial \mathcal{L}[Y(u), Y_{n,\bar{m}}(u)]}{\partial Y^\mu(u)}\delta(u-z'')\Big) + \Big(z' \leftrightarrow z''\Big) \\
&= \; \eta^{\mu\nu}\Big(-\frac{\partial}{\partial x'_1}[\delta(z''-z')ik_0.iK_{1;0}\mathcal{L}[z'']]\Big) + \Big(z' \leftrightarrow z''\Big) \\
&= \eta^{\mu\nu}\Big(-[\frac{\partial}{\partial x'_1} + \frac{\partial}{\partial x''_1}][\delta(z''-z')ik_0.iK_{1;0}\mathcal{L}[z'']]\Big)
\end{aligned}
$$

We restore the integrals over z', z'', and use $G(z',z'') = \langle Y(z')Y(z'')\rangle$ and integrate by parts on x', x'' to get

$$
\begin{aligned}
&\frac{d}{dln\;a}\int dz'dz'' \; (\frac{\partial}{\partial x'_1} + \frac{\partial}{\partial x''_1})\langle Y(z')Y(z'')\rangle[\delta(z''-z')ik_0.iK_{1;0}\mathcal{L}[z'']] = \\
&= \frac{d}{dln\;a}\int dz' \; [\frac{\partial}{\partial x'_1}\langle Y(z')Y(z')\rangle]ik_0.iK_{1;0}\mathcal{L}[z']]
\end{aligned}
$$

$$= -\int dz'\ \dot{G}(z',z') ik_0.iK_{1;0}\frac{\partial}{\partial x_1'}[\mathcal{L}[z']]$$

We have integrated by parts again in the last step.

Finally we can add the complex conjugate to obtain:

$$= \int dz'\ \dot{G}(z',z')\Big(k_0.K_{1;0}\frac{\partial}{\partial x_1'}[\mathcal{L}[z']] + k_0.K_{0;\bar{1}}\frac{\partial}{\partial \bar{x}_1'}[\mathcal{L}[z']]\Big) \tag{B.1.3}$$

3.

$$\eta^{\mu\nu}\int dz'\int dz''\ \dot{G}(z',z'')\int du\ [\frac{\partial}{\partial x_1}\delta(u-z')][\frac{\partial}{\partial x_1}\delta(u-z'')]\frac{\partial^2\mathcal{L}[u]}{\partial Y^\mu_{1;0}(u)\partial Y^\nu_{1;0}(u)}$$

$$= \frac{d}{dln\ a}\eta^{\mu\nu}\int dz'\int dz''\langle Y_{1;0}(z')Y_{1;0}(z'')\rangle\delta(z'-z'')\frac{\partial^2\mathcal{L}[z']}{\partial Y^\mu_{1;0}(z')\partial Y^\nu_{1;0}(z')}$$

$$= \frac{d}{dln\ a}\eta^{\mu\nu}\int dz'[\frac{1}{2}(\frac{\partial^2}{\partial x_1'^2} - \frac{\partial}{\partial x_2'})\langle Y(z')Y(z')\rangle]\frac{\partial^2\mathcal{L}[z']}{\partial Y^\mu_{1;0}(z')\partial Y^\nu_{1;0}(z')}$$

$$= \eta^{\mu\nu}\int dz'\dot{G}(z',z')\frac{1}{2}(\frac{\partial^2}{\partial x_1'^2} + \frac{\partial}{\partial x_2'})\frac{\partial^2\mathcal{L}[z']}{\partial Y^\mu_{1;0}(z')\partial Y^\nu_{1;0}(z')}$$

$$= -\int dz'\dot{G}(z',z')K_{1;0}.K_{1;0}\frac{1}{2}(\frac{\partial^2}{\partial x_1'^2} + \frac{\partial}{\partial x_2'})\mathcal{L}[z'] \tag{B.1.4}$$

4. The complex conjugate is:

$$-\int dz'\dot{G}(z',z')K_{0;\bar{1}}.K_{0;\bar{1}}\frac{1}{2}(\frac{\partial^2}{\partial \bar{x}_1'^2} + \frac{\partial}{\partial \bar{x}_2'})\mathcal{L}[z'] \tag{B.1.5}$$

5.

$$\int dz'dz''\ \dot{G}(z',z'')\Big(\eta^{\mu\nu}\int du\ [\frac{\partial}{\partial x_2}\delta(u-z')]\delta(u-z'')\frac{\partial^2\mathcal{L}[u]}{\partial Y^\mu_{2;0}(u)\partial Y^\nu(u)}\ +\ z'\leftrightarrow z''\Big)$$

$$= \int dz'dz''\ \dot{G}(z',z'')\Big(\eta^{\mu\nu}\ [-\frac{\partial}{\partial x_2'}\delta(z''-z')]\frac{\partial^2\mathcal{L}[z'']}{\partial Y^\mu_{2;0}(z'')\partial Y^\nu(z'')}\ +\ z'\leftrightarrow z''\Big)$$

$$= \int dz'dz''\ \frac{d}{dln\ a}[\frac{\partial}{\partial x_2'} + \frac{\partial}{\partial x_2''}]G(z',z'')\Big(\eta^{\mu\nu}\ [\delta(z''-z')]\frac{\partial^2\mathcal{L}[z'']}{\partial Y^\mu_{2;0}(z'')\partial Y^\nu(z'')}\Big)$$

$$= \int dz'\ \dot{G}(z',z')\Big(K_{2;0}.k_0\ (\frac{\partial}{\partial x_2}\mathcal{L}[z''])\Big) \tag{B.1.6}$$

6. Complex conjugate gives:

$$= \int dz'\ \dot{G}(z',z')\Big(K_{0;\bar{2}}.k_0\ (\frac{\partial}{\partial \bar{x}_2}\mathcal{L}[z''])\Big) \tag{B.1.7}$$

7.

$$\int dz'dz''\ \dot{G}(z',z'')\Big(\eta^{\mu\nu}\int du\ [\frac{\partial^2}{\partial x_1\partial \bar{x}_1}\delta(u-z')]\delta(u-z'')\frac{\partial^2\mathcal{L}[u]}{\partial Y^\mu_{1;\bar{1}}(u)\partial Y^\nu(u)}\ +\ z'\leftrightarrow z''\Big)$$

$$= \frac{d}{d\ ln\ a}\int dz'dz''\ \langle Y(z')_{1;\bar{1}}Y(z'') + Y(z')Y_{1;\bar{1}}(z'')\rangle\Big(\eta^{\mu\nu}\delta(z'-z'')\frac{\partial^2\mathcal{L}[z']}{\partial Y^\mu_{1;\bar{1}}(z')\partial Y^\nu(z')}\Big)$$

$$= \frac{d}{d\ ln\ a}\int dz'dz''\ \langle Y(z')_{1;\bar{1}}Y(z'') + Y(z')Y_{1;\bar{1}}(z'')\rangle\Big((iK_{1;\bar{1}}.ik_0)\delta(z'-z'')\mathcal{L}[z']\Big) \tag{B.1.8}$$

8.

$$\int dz'dz''\ \dot{G}(z',z'')\Big(\eta^{\mu\nu}\int du\ [\frac{\partial}{\partial\bar{x}_1}\delta(u-z')][\frac{\partial}{\partial x_1}\delta(u-z'')]\frac{\partial^2\mathcal{L}[u]}{\partial Y^{\mu}_{1;0}(u)\partial Y^{\nu}_{0;\bar{1}}(u)}\ +\ z'\leftrightarrow z''\Big)$$

$$=\frac{d}{d\ ln\ a}\int dz'dz''\ \langle Y_{0:\bar{1}}(z')Y_{1;0}(z'')+Y_{0:\bar{1}}(z'')Y_{1;0}(z')\rangle\Big(\delta(z'-z'')(iK_{1;0}.iK_{0;\bar{1}})\mathcal{L}[z']\Big) \tag{B.1.9}$$

(B.1.8) and (B.1.9) can be added to give

$$=\frac{d}{d\ ln\ a}\int dz'\ [\frac{\partial^2}{\partial x'_1\partial\bar{x}'_1}\langle Y(z')Y(z')\rangle]\Big((iK_{1;0}.iK_{0;\bar{1}})\mathcal{L}[z']\Big)$$

$$=\int dz'\ \dot{G}(z',z')\Big((iK_{1;0}.iK_{0;\bar{1}})[\frac{\partial^2}{\partial x'_1\partial\bar{x}'_1}\mathcal{L}[z']]\Big) \tag{B.1.10}$$

provided the following constraint is imposed :

$$K_{1;0}.K_{0;\bar{1}}\ \mathcal{L}=K_{1;\bar{1}}.k_0\ \mathcal{L} \tag{B.1.11}$$

The constraint is gauge covariant since both sides have identical gauge transformation properties. Since $K_{1;\bar{1}}$ is an auxiliary field (i.e. not physical) we are free to impose this constraint. In fact since $K_{1;\bar{1}}.k_0$ contains $q_{1;\bar{1}}q_0$ (for $q_0\neq 0$) , this can be treated as an algebraic constraint on $q_{1;\bar{1}}$.

The massless case (Graviton) is discussed in Section 3 and Section 4.

Similar constraints on $K_{n;\bar{m}}$ occur at every level. We will refer to them as K-constraints. They are described in the next Appendix (D).

B.2 Level (1,1)

The terms calculated above are sufficient to extract the coefficient of the graviton multiplet vertex operators at level $(1;\bar{1})$, $Y^{\mu}_{1;0}Y^{\nu}_{0;\bar{1}}$ and the next massive level, $Y^{\mu}_{1;0}Y^{\nu}_{1;0}Y^{\rho}_{0;\bar{1}}Y^{\sigma}_{0;\bar{1}}e^{ik_0Y}$, a vertex operator in closed string theory at level $(2;\bar{2})$. We revert to the notation $k_1=K_{1;0},k_{\bar{1}}=K_{0;\bar{1}},...$ below.

We get for level $(1,\bar{1})$:

$$[-k_0^2k_{1\mu}k_{\bar{1}\nu}+k_0.k_1k_{0\mu}k_{\bar{1}\nu}+k_0.k_{\bar{1}}k_{0\nu}k_{1\mu}-k_1.k_{\bar{1}}k_{0\mu}k_{0\nu}]Y^{\mu}_{1;0}Y^{\nu}_{0;\bar{1}}=0 \tag{B.2.12}$$

$$[-k_0^2K_{1;\bar{1}\mu}+k_0.k_1k_{\bar{1}\mu}+k_0.k_{\bar{1}}k_{1\mu}-k_1.k_{\bar{1}}k_{0\mu}]Y^{\mu}_{1;\bar{1}}=0 \tag{B.2.13}$$

If we use the constraint (B.1.11), then the two equations above are not independent: $k_{0\mu}$ dotted into (B.2.13) is equal to the trace of (B.2.12).

Furthermore we can dimensionally reduce (B.2.13) to get an equation that looks exactly like (B.2.13) but with the dot product going over D dimensions. The remaining term in the dot product cancels if we use the definition

$$K_{1;\bar{1}\mu}=\frac{q_1}{q_0}k_{\bar{1}\mu}+\frac{q_{\bar{1}}}{q_0}k_{1\mu}-\frac{q_1q_{\bar{1}}}{q_0^2}k_{0\mu} \tag{B.2.14}$$

Note that according to this definition, when $\mu=D$, $Q_{1;\bar{1}}$ is fixed to:

$$q_0Q_{1;\bar{1}}=q_1\bar{q}_1$$

Thus $q_0Q_{1;\bar{1}}$ also stands for the dilaton. For the other values of the indices the situation is more complicated. In the closed string there is the extra condition that the number of q_1's must equal the number of $\bar{q}_1$'s in any term - this is the analog of the condition that there be no q_1 in the open string expressions. This means that the first two terms in the definition of $K_{1;\bar{1}\mu}$ are not allowed - either they have to be set to zero, or set equal to some other operator in a way that is consistent with gauge transformation. $q_1\bar{q}_1$ is the dilaton. Thus we can think of (B.2.13) as a way of defining $K_{1;\bar{1}\mu}$ in terms of the other physical fields.

Now another subtlety here is that $q_0=0$ at this (massless) level. Thus it is not clear what to make of (B.2.14). Since this variable does not occur in the graviton equation, after using the K-constraint, we will not worry about this problem. At higher level this variable occurs in conjunction with other k_n,q_n so we can apply the Q-rules to get rid of them. Also $q_0\neq 0$ at higher levels. This problem occurs only for level 1.

B.3 Level (2,2)

At level $(2,\bar{2})$ for $Y^{\mu}_{1;0}Y^{\nu}_{1;0}Y^{\rho}_{0;\bar{1}}Y^{\sigma}_{0;\bar{1}}e^{ik_0Y}$ we get:

$$-\frac{1}{4}k_0^2(k_1.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_1(k_0.Y_1)(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}})^2+\frac{1}{2}k_0.k_{\bar{1}}(k_0.Y_{\bar{1}})(k_{\bar{1}}.Y_{\bar{1}})(k_1.Y_1)^2$$
$$-\frac{k_1.k_1}{4}(k_0.Y_1)^2(k_{\bar{1}}.Y_{\bar{1}})^2-\frac{k_{\bar{1}}.k_{\bar{1}}}{4}(k_0.Y_{\bar{1}})^2(k_1.Y_1)^2-k_1.k_{\bar{1}}(k_0.Y_1)(k_0.Y_{\bar{1}})(k_1.Y_1)(k_{\bar{1}}.Y_{\bar{1}}) \qquad \text{(B.3.15)}$$

This can easily be seen to be gauge invariant under $k_{1\mu}\to k_{1\mu}+\lambda_1 k_{0\mu}$ and $k_{\bar{1}\mu}\to k_{\bar{1}\mu}+\lambda_{\bar{1}}k_{0\mu}$, after using the tracelessness condition on the gauge parameter, $\lambda_1 k_1.k_{\bar{1}}=0=\lambda_1 k_{\bar{1}}.k_{\bar{1}}$ and the same for its complex conjugate.

Appendix C Q-rules for Level 5

We use the notation $QQ[...]$ for the Q-rules.

Four Index

$$\begin{aligned}
QQ\left[k_\mu k_{1\nu}k_{1\rho}k_{1\sigma}q_1\right] &= \frac{1}{4}\left(k_{2\mu}k_{1\nu}k_{1\rho}k_{1\sigma}+k_{1\mu}k_{2\nu}k_{1\rho}k_{1\sigma}+k_{1\mu}k_{1\nu}k_{2\rho}k_{1\sigma}+k_{1\mu}k_{1\nu}k_{1\rho}k_{2\sigma}\right)q_0\\
QQ\left[k_{1\mu}k_{1\nu}k_{1\rho}q_1\lambda_1\right] &= \frac{1}{4}q_0\left(\left(k_{2\mu}k_{1\nu}k_{1\rho}+k_{1\mu}k_{2\nu}k_{1\rho}+k_{1\mu}k_{1\nu}k_{2\rho}\right)\lambda_1+k_{1\mu}k_{1\nu}k_{1\rho}\lambda_2\right)
\end{aligned} \qquad \text{(C.0.1)}$$

Three Index

$$\begin{aligned}
QQ\left[k_{2\mu}k_{1\nu}k_{1\rho}q_1\right] &= b_3k_{3\mu}k_{1\nu}k_{1\rho}q_0+a_3k_{1\mu}k_{2\nu}k_{2\rho}q_0+\frac{1}{2}a_{3p}k_{2\mu}\left(k_{2\nu}k_{1\rho}+k_{1\nu}k_{2\rho}\right)q_0\\
&+ \frac{1}{2}b_{3p}k_{1\mu}\left(k_{3\nu}k_{1\rho}+k_{1\nu}k_{3\rho}\right)q_0+c_3k_{1\mu}k_{1\nu}k_{1\rho}q_2\\
QQ\left[k_{1\nu}k_{1\rho}q_1\lambda_2\right] &= \frac{1}{2}b_{3p}k_{3\nu}k_{1\rho}q_0\lambda_1+a_3k_{2\nu}k_{2\rho}q_0\lambda_1+\frac{1}{2}b_{3p}k_{1\nu}k_{3\rho}q_0\lambda_1+c_3k_{1\nu}k_{1\rho}q_2\lambda_1\\
&+ \frac{1}{2}a_{3p}k_{2\nu}k_{1\rho}q_0\lambda_2+\frac{1}{2}a_{3p}k_{1\nu}k_{2\rho}q_0\lambda_2+b_3k_{1\nu}k_{1\rho}q_0\lambda_3\\
QQ\left[k_{2\mu}k_{1\rho}q_1\lambda_1\right] &= b_3k_{3\mu}k_{1\rho}q_0\lambda_1+\frac{1}{2}a_{3p}k_{2\mu}k_{2\rho}q_0\lambda_1+\frac{1}{2}b_{3p}k_{1\mu}k_{3\rho}q_0\lambda_1+c_3k_{1\mu}k_{1\rho}q_2\lambda_1+\frac{1}{2}a_{3p}k_{2\mu}k_{1\rho}q_0\lambda_2\\
&+ a_3k_{1\mu}k_{2\rho}q_0\lambda_2+\frac{1}{2}b_{3p}k_{1\mu}k_{1\rho}q_0\lambda_3
\end{aligned} \qquad \text{(C.0.2)}$$

Three Index 2 q's

$$\begin{aligned}
QQ\left[k_{1\mu}k_{1\nu}k_{1\rho}q_1^2\right] &= b_{32}\left(k_{2\mu}k_{2\nu}k_{1\rho}+k_{2\mu}k_{1\nu}k_{2\rho}+k_{1\mu}k_{2\nu}k_{2\rho}\right)q_0^2+c_{32}\left(k_{3\mu}k_{1\nu}k_{1\rho}+k_{1\mu}k_{3\nu}k_{1\rho}+k_{1\mu}k_{1\nu}k_{3\rho}\right)q_0^2\\
&+ a_{32}k_{1\mu}k_{1\nu}k_{1\rho}q_0q_2\\
QQ\left[k_{1\mu}k_{1\nu}k_{1\rho}q_1\lambda_1\right] &= \frac{1}{4}q_0\left(\left(k_{2\mu}k_{1\nu}k_{1\rho}+k_{1\mu}k_{2\nu}k_{1\rho}+k_{1\mu}k_{1\nu}k_{2\rho}\right)\lambda_1+k_{1\mu}k_{1\nu}k_{1\rho}\lambda_2\right)
\end{aligned} \qquad \text{(C.0.3)}$$

Two index

$$
\begin{aligned}
QQ\left[k_{2\mu}k_{2\nu}q_1\right] &= \frac{1}{2}d_2\left(k_{3\mu}k_{2\nu}+k_{2\mu}k_{3\nu}\right)q_0 \\
&+ \frac{1}{2}c_2\left(k_{4\mu}k_{1\nu}+k_{1\mu}k_{4\nu}\right)q_0+\frac{1}{2}b_2\left(k_{2\mu}k_{1\nu}+k_{1\mu}k_{2\nu}\right)q_2+a_2k_{1\mu}k_{1\nu}q_3 \\
QQ\left[k_{3\mu}k_{1\nu}q_1\right] &= D_{2a}\left(k_{3\mu}k_{2\nu}-k_{2\mu}k_{3\nu}\right)q_0+D_{2s}\left(k_{3\mu}k_{2\nu}+k_{2\mu}k_{3\nu}\right)q_0+C_{2a}\left(k_{4\mu}k_{1\nu}-k_{1\mu}k_{4\nu}\right)q_0 \\
&+ C_{2s}\left(k_{4\mu}k_{1\nu}+k_{1\mu}k_{4\nu}\right)q_0+B_{2a}\left(k_{2\mu}k_{1\nu}-k_{1\mu}k_{2\nu}\right)q_2+B_{2s}\left(k_{2\mu}k_{1\nu}+k_{1\mu}k_{2\nu}\right)q_2+A_{2p}k_{1\mu}k_{1\nu}q_3 \\
QQ\left[k_{2\nu}q_1\lambda_2\right] &= \frac{1}{2}c_2k_{4\nu}q_0\lambda_1+\frac{1}{2}b_2k_{2\nu}q_2\lambda_1+a_2k_{1\nu}q_3\lambda_1+\frac{1}{2}d_2k_{3\nu}q_0\lambda_2+\frac{1}{2}b_2k_{1\nu}q_2\lambda_2+\frac{1}{2}d_2k_{2\nu}q_0\lambda_3+\frac{1}{2}c_2k_{1\nu}q_0\lambda_4 \\
QQ\left[k_{1\nu}q_1\lambda_3\right] &= -C_{2a}k_{4\nu}q_0\lambda_1+C_{2s}k_{4\nu}q_0\lambda_1-B_{2a}k_{2\nu}q_2\lambda_1+B_{2s}k_{2\nu}q_2\lambda_1+A_{2p}k_{1\nu}q_3\lambda_1-D_{2a}k_{3\nu}q_0\lambda_2 \\
&+ D_{2s}k_{3\nu}q_0\lambda_2+B_{2a}k_{1\nu}q_2\lambda_2+B_{2s}k_{1\nu}q_2\lambda_2+D_{2a}k_{2\nu}q_0\lambda_3+D_{2s}k_{2\nu}q_0\lambda_3+C_{2a}k_{1\nu}q_0\lambda_4+C_{2s}k_{1\nu}q_0\lambda_4 \\
QQ\left[k_{3\mu}q_1\lambda_1\right] &= C_{2a}k_{4\mu}q_0\lambda_1+C_{2s}k_{4\mu}q_0\lambda_1+B_{2a}k_{2\mu}q_2\lambda_1+B_{2s}k_{2\mu}q_2\lambda_1+A_{2p}k_{1\mu}q_3\lambda_1+D_{2a}k_{3\mu}q_0\lambda_2 \\
&+ D_{2s}k_{3\mu}q_0\lambda_2-B_{2a}k_{1\mu}q_2\lambda_2+B_{2s}k_{1\mu}q_2\lambda_2-D_{2a}k_{2\mu}q_0\lambda_3+D_{2s}k_{2\mu}q_0\lambda_3-C_{2a}k_{1\mu}q_0\lambda_4+C_{2s}k_{1\mu}q_0\lambda_4
\end{aligned}
\tag{C.0.4}
$$

Two index with q^2

$$
QQ\left[q_1q_2k_{1,\mu}k_{1,\nu}\right] = d_{22}\left(k_{3\mu}k_{2\nu}+k_{2\mu}k_{3\nu}\right)q_0^2+a_{22}q_0q_3k_{1,\mu}k_{1,\nu}+b_{22}q_0q_2\left(k_{2\nu}k_{1,\mu}+k_{2\mu}k_{1,\nu}\right)+c_{22}q_0^2\left(k_{4\nu}k_{1,\mu}+k_{4\mu}k_{1,\nu}\right)
\tag{C.0.5}
$$

$$
\begin{aligned}
QQ[k_{2\mu}q_1^2k_{1,\nu}] &= q_0[d_{23a}(k_{3\mu}k_{2\nu}-k_{2\mu}k_{3\nu})q_0+d_{23s}(k_{3\mu}k_{2\nu}+k_{2\mu}k_{3\nu}q_0+a_{23}q_3k_{1,\mu}k_{1,\nu} \\
&+ b_{23a}q_2(-k_{2\nu}k_{1,\mu}+k_{2\mu}k_{1,\nu})+b_{23s}q_2(k_{2\nu}k_{1,\mu}+k_{2\mu}k_{1,\nu})+c_{23a}q_0(-k_{4\nu}k_{1,\mu}+k_{4\mu}k_{1,\nu}) \\
&+ c_{23s}q_0(k_{4\nu}k_{1,\mu}+k_{4\mu}k_{1,\nu})]
\end{aligned}
\tag{C.0.6}
$$

$$
\begin{aligned}
QQ\left[q_1^2\lambda_1k_{1,\mu}k_{1,\nu}\right] &= b_{32}k_{2\mu}k_{2\nu}q_0^2\lambda_1+c_{32}k_{3\nu}q_0^2\lambda_1k_{1,\mu}+b_{32}k_{2\nu}q_0^2\lambda_2k_{1,\mu}+c_{32}k_{3\mu}q_0^2\lambda_1k_{1,\nu}+b_{32}k_{2\mu}q_0^2\lambda_2k_{1,\nu} \\
&+ a_{32}q_0q_2\lambda_1k_{1,\mu}k_{1,\nu}+c_{32}q_0^2\lambda_3k_{1,\mu}k_{1,\nu} \\
QQ\left[q_1^2\lambda_2k_{1,\nu}\right] &= q_0(-c_{23a}k_{4\nu}q_0\lambda_1+c_{23s}k_{4\nu}q_0\lambda_1-b_{23a}k_{2\nu}q_2\lambda_1+b_{23s}k_{2\nu}q_2\lambda_1-d_{23a}k_{3\nu}q_0\lambda_2+d_{23s}k_{3\nu}q_0\lambda_2 \\
&+ d_{23a}k_{2\nu}q_0\lambda_3+d_{23s}k_{2\nu}q_0\lambda_3+a_{23}q_3\lambda_1k_{1,\nu}+b_{23a}q_2\lambda_2k_{1,\nu}+b_{23s}q_2\lambda_2k_{1,\nu}+c_{23a}q_0\lambda_4k_{1,\nu} \\
&+ c_{23s}q_0\lambda_4k_{1,\nu}) \\
QQ[k_{2\mu}q_1^2\lambda_1] &= q_0(c_{23a}k_{4\mu}q_0\lambda_1+c_{23s}k_{4\mu}q_0\lambda_1+b_{23a}k_{2\mu}q_2\lambda_1+b_{23s}k_{2\mu}q_2\lambda_1+d_{23a}k_{3\mu}q_0\lambda_2+d_{23s}k_{3\mu}q_0\lambda_2 \\
&- d_{23a}k_{2\mu}q_0\lambda_3+d_{23s}k_{2\mu}q_0\lambda_3+a_{23}q_3\lambda_1k_{1,\mu}-b_{23a}q_2\lambda_2k_{1,\mu}+b_{23s}q_2\lambda_2k_{1,\mu}-c_{23a}q_0\lambda_4k_{1,\mu} \\
&+ c_{23s}q_0\lambda_4k_{1,\mu})
\end{aligned}
\tag{C.0.7}
$$

Two index q^3

$$
\begin{aligned}
QQ\left[q_1^3k_{1,\mu}k_{1,\nu}\right] &= d_{24}\left(k_{3\mu}k_{2\nu}+k_{2\mu}k_{3\nu}\right)q_0^3+a_{24}q_0^2q_3k_{1,\mu}k_{1,\nu}+b_{24}q_0^2q_2\left(k_{2\nu}k_{1,\mu}+k_{2\mu}k_{1,\nu}\right)+c_{24}q_0^3\left(k_{4\nu}k_{1,\mu}+k_{4\mu}k_{1,\nu}\right) \\
QQ\left[q_1^3\lambda_1k_{1,\nu}\right] &= c_{24}k_{4\nu}q_0^3\lambda_1+b_{24}k_{2\nu}q_0^2q_2\lambda_1+d_{24}k_{3\nu}q_0^3\lambda_2+d_{24}k_{2\nu}q_0^3\lambda_3+a_{24}q_0^2q_3\lambda_1k_{1,\nu}+b_{24}q_0^2q_2\lambda_2k_{1,\nu} \\
&+ c_{24}q_0^3\lambda_4k_{1,\nu}
\end{aligned}
\tag{C.0.8}
$$

One index

$$
\begin{aligned}
QQ\left[k_{4\mu}q_0q_1\right] &= a_1k_{5,m}q_0^2 + b1k_{3\mu}q_0q_2 + c1k_{2\mu}q_0q_3 + d1q_4q_0k_{1,\mu} + e1q_2^2k_{1,\mu} \\
QQ\left[k_{3\mu}q_1^2\right] &= a_{11}1k_{5,m}q_0^2 + b_{11}k_{3\mu}q_0q_2 + c_{11}k_{2\mu}q_0q_3 + d_{11}q_4q_0k_{1,\mu} + e_{11}q_2^2k_{1,\mu} \\
QQ\left[q_1q_3k_{1,\mu}\right] &= a_{13}k_{5,m}q_0^2 + b_{13}k_{3\mu}q_0q_2 + c_{13}k_{2\mu}q_0q_3 + d_{13}q_4q_0k_{1,\mu} + e_{13}q_2^2k_{1,\mu} \\
QQ\left[k_{2\mu}q_1^3\right] &= a_{14}k_{5\mu}q_0^3 + b_{14}k_{3\mu}q_0^2q_2 + c_{14}k_{2\mu}q_0^2q_3 + e_{14}q_0q_2^2k_{1,\mu} + d_{14}q_0^2q_4k_{1,\mu} \\
QQ\left[q_1^4k_{1,\mu}\right] &= a_{16}k_{5\mu}q_0^4 + b_{16}k_{3\mu}q_0^3q_2 + c_{16}k_{2\mu}q_0^3q_3 + e_{16}q_0^2q_2^2k_{1,\mu} + d_{16}q_0^3q_4k_{1,\mu} \\
QQ\left[q_1^4\lambda_1\right] &= e_{16}q_0^2q_2^2\lambda_1 + d_{16}q_0^3q_4\lambda_1 + c_{16}q_0^3q_3\lambda_2 + b_{16}q_0^3q_2\lambda_3 + a_{16}q_0^4\lambda_5 \\
QQ\left[q_1q_2\lambda_2\right] &= e_{12}q_2^2\lambda_1 + d_{12}q_0q_4\lambda_1 + c_{12}q_0q_3\lambda_2 + b_{12}q_0q_2\lambda_3 + a12q_0^2\lambda_5 \\
QQ\left[q_0q_1\lambda_4\right] &= e1q_2^2\lambda_1 + d1q_0q_4\lambda_1 + c1q_0q_3\lambda_2 + b1q_0q_2\lambda_3 + a1q_0^2\lambda_5 \\
QQ\left[q_1^2\lambda_3\right] &= e_{11}q_2^2\lambda_1 + d_{11}q_0q_4\lambda_1 + c_{11}q_0q_3\lambda_2 + b_{11}q_0q_2\lambda_3 + a_{11}q_0^2\lambda_5 \\
QQ\left[q_1q_3\lambda_1\right] &= e_{13}q_2^2\lambda_1 + d_{13}q_0q_4\lambda_1 + c_{13}q_0q_3\lambda_2 + b_{13}q_0q_2\lambda_3 + a_{13}q_0^2\lambda_5 \\
QQ\left[q_1^3\lambda_2\right] &= e_{14}q_0q_2^2\lambda_1 + d_{14}q_0^2q_4\lambda_1 + c_{14}q_0^2q_3\lambda_2 + b_{14}q_0^2q_2\lambda_3 + a_{14}q_0^3\lambda_5 \\
QQ\left[q_1^2q_2\lambda_1\right] &= e_{15}q_0q_2^2\lambda_1 + d_{15}q_0^2q_4\lambda_1 + c_{15}q_0^2q_3\lambda_2 + b_{15}q_0^2q_2\lambda_3 + a_{15}q_0^3\lambda_5
\end{aligned} \tag{C.0.9}
$$

No index

$$
\begin{aligned}
QQ\left[q_1q_4\right] &= B_1q_2q_3 + A_1q_0q_5 \\
QQ\left[q_1q_2^2\right] &= B_2q_0q_2q_3 + A2q_0^2q_5 \\
QQ\left[q_1^2q_3\right] &= B_3q_0q_2q_3 + A3q_0^2q_5 \\
QQ\left[q_1^3q_2\right] &= B_4q_0^2q_2q_3 + A4q_0^3q_5 \\
QQ\left[q_1^5\right] &= B_5q_0^3q_2q_3 + A_5q_0^4q_5
\end{aligned} \tag{C.0.10}
$$

Final solution in terms of $b_2, C_{2a}, c_{23a}, b_{22}$

$$\left\{a_{3p} = -\frac{2}{5}(-2+b_3), b_{3p} = 2(-1+b_3), c_3 = -\frac{3}{5}(-3+4b_3), a_3 = \frac{2-b_3}{5}\right\}$$

$$\left\{c_2 = \frac{1}{4}(-2-5b_2), d_2 = \frac{6-b_2}{4}, a_2 = \frac{b_2}{2}, b_3 = \frac{-6+7b_2}{-8+6b_2}\right\}$$

$$\left\{B_{2a} = \frac{1}{2} - C_{2a}, D_{2a} = \frac{1}{2}, C_{2s} = \frac{1}{8}(2-5b_2), D_{2s} = \frac{2-b_2}{8}, A_{2p} = \frac{b_2}{2}, B_{2s} = \frac{b_2}{2}\right\}$$

$$\left\{a_{32} = \frac{2}{5} - \frac{12c_{32}}{5}, b_{32} = \frac{1}{5} - \frac{c_{32}}{5}\right\}$$

$$\left\{c_{22} = \frac{1}{8} - \frac{5b_{22}}{4}, d_{22} = \frac{1}{8} - \frac{b_{22}}{4}, c_{32} = -\frac{3-b_2}{2(-4+3b_2)} + \frac{5b_{22}}{-4+3b_2}, a_{22}a_{22} = \frac{1}{2} + b_{22}\right\}$$

$$\left\{b_{23a} = \frac{1}{2} - c_{23a}, d_{23a} = \frac{1}{2}, d_{23s} = \frac{1}{8}(3-b_2-2b_{22})\right\}$$

$$\left\{c_{23s} = \frac{1}{8}(3-5b_2-10b_{22}), a_{23} = \frac{1}{2}(-1+b_2+2b_{22}), b_{23s} = \frac{1}{2}(b_2+2b_{22})\right\}$$

$$\left\{c_{24} = \frac{1}{8}(7-30b_{22}), d_{24} = \frac{3}{8} - \frac{3b_{22}}{4}, a_{24} = -\frac{1}{2} + 3b_{22}, b_{24} = -\frac{1}{2} + 3b_{22}\right\}$$

$$\left\{a_1 = \frac{10+8b_{22}(3-2C_{2a})+4C_{2a}-3b_2(5+2C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}, b1 = \frac{(1+2b_{22})(-3+2C_{2a})+b_2(3+6C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}\right\}$$

$$\left\{c1=\frac{1+9b_2-6b_{22}+2C_{2a}+6b_2C_{2a}+4b_{22}C_{2a}}{7+3b_2+6b_{22}+6C_{2a}+6b_2C_{2a}-4b_{22}C_{2a}},d1=\frac{-7+2C_{2a}+b_{22}(-6+4C_{2a})+b_2(-3+6C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}\right\}$$

$$e1=-\frac{(2+3b_2)(-3+2C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}$$

$$\left\{a_{11}=\frac{4(2+4C_{2a}-3b_2(1+2C_{2a})+b_{22}(-6+4C_{2a}))}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})},b_{11}=\frac{2(3+3b_2+6b_{22}-2C_{2a}+6b_2C_{2a}-4b_{22}C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}\right\}$$

$$\left\{c_{11}=\frac{2(-1+b_{22}(6-4C_{2a})-2C_{2a}+b_2(3+6C_{2a}))}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})},d_{11}=-\frac{16C_{2a}}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}\right\}$$

$$e_{11}=\frac{-5+b_{22}(6-4C_{2a})+14C_{2a}+b_2(3+6C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}$$

$$a_{12}=\frac{(2+8c_{23a}+8b_{22}(-3+6c_{23a}-8C_{2a})-3b_2(1+4c_{23a}-2C_{2a})-4C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$b_{12}=\frac{(-(1+2b_{22})(-5+8c_{23a}-10C_{2a})+b_2(3+6C_{2a}))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$c_{12}=\frac{(3-2c_{23a}+b_2(3+6c_{23a})+4C_{2a}+b_{22}(14-20c_{23a}+24C_{2a}))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$d_{12}=\frac{(-2(7+3b_2+6b_{22})c_{23a}+2(5+3b_2+10b_{22})C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$e_{12}=\frac{(-3+6b_{22}+16c_{23a}+12b_2c_{23a}-2(7+6b_2+2b_{22})C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$a_{13}=\frac{(6+3b_2(-3+4c_{23a}-2C_{2a})+4C_{2a}-8(c_{23a}+6b_{22}c_{23a}+2b_{22}C_{2a}))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$b_{13}=\frac{-6+8c_{23a}-4C_{2a}+4b_{22}(-1+4c_{23a}+2C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}$$

$$c_{13}=\frac{(2(1+c_{23a}+C_{2a}+3b_2(1-c_{23a}+C_{2a})+2b_{22}(1+5c_{23a}+C_{2a})))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$d_{13}=\frac{2((7+3b_2+6b_{22})c_{23a}+(1+3b_2+2b_{22})C_{2a})}{7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a})}$$

$$e_{13}=\frac{(5+6b_2+6b_{22}-16c_{23a}-12b_2c_{23a}+2C_{2a}-4b_{22}C_{2a})}{(7+3b_2+6b_{22}+6C_{2a}+6b_2C_{2a}-4b_{22}C_{2a})}$$

$$a_{14}=\frac{(13+24c_{23a}+2b_{22}(-33+72c_{23a}-74C_{2a})-6b_2(1+6c_{23a}-2C_{2a})+2C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$b_{14}=\frac{(2(b_2(3+6C_{2a})+4(1-3c_{23a}+2C_{2a}+b_{22}(3-6c_{23a}+6C_{2a}))))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$c_{14}=\frac{(-5-6c_{23a}+3b_2(1+6c_{23a}-2C_{2a})-2C_{2a}+30b_{22}(1-2c_{23a}+2C_{2a}))}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$d_{14}=\frac{(-6(7+3b_2+6b_{22})c_{23a}+4(4+3b_2+12b_{22})C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$e_{14}=\frac{(-9+18b_{22}+12(4+3b_2)c_{23a}-2(13+12b_2+6b_{22})C_{2a})}{(7+b_{22}(6-4C_{2a})+6C_{2a}+b_2(3+6C_{2a}))}$$

$$a_{15} = \frac{(11 - 8c_{23a} + 3b_2(-1 + 4c_{23a} - 2C_{2a}) + 14C_{2a} - 6b_{22}(7 + 8c_{23a} + 6C_{2a}))}{(7 + b_{22}(6 - 4C_{2a}) + 6C_{2a} + b_2(3 + 6C_{2a}))}$$

$$b_{15} = \frac{4(-1 + 2c_{23a} - 2C_{2a} + b_{22}(2 + 4c_{23a} + 4C_{2a}))}{7 + b_{22}(6 - 4C_{2a}) + 6C_{2a} + b_2(3 + 6C_{2a})}$$

$$c_{15} = \frac{(-1 + 2c_{23a} - 2C_{2a} + b_2(3 - 6c_{23a} + 6C_{2a}) + 2b_{22}(11 + 10c_{23a} + 6C_{2a}))}{(7 + b_{22}(6 - 4C_{2a}) + 6C_{2a} + b_2(3 + 6C_{2a}))}$$

$$d_{15} = \frac{2((7 + 3b_2 + 6b_{22})c_{23a} - 4C_{2a} + 8b_{22}C_{2a})}{7 + b_{22}(6 - 4C_{2a}) + 6C_{2a} + b_2(3 + 6C_{2a})}$$

$$e_{15} = \frac{(1 - 16c_{23a} + 6b_{22}(3 - 2C_{2a}) + 10C_{2a} + b_2(3 - 12c_{23a} + 6C_{2a}))}{(7 + b_{22}(6 - 4C_{2a}) + 6C_{2a} + b_2(3 + 6C_{2a}))}$$

$$a_{16} = \frac{-(25 + 3b_2 - 102b_{22} + 26C_{2a} + 6b_2C_{2a} - 124b_{22}C_{2a})}{(-7 - 3b_2 - 6b_{22} - 6C_{2a} - 6b_2C_{2a} + 4b_{22}C_{2a})}$$

$$\left\{b_{16} = \frac{4(-1 + 6b_{22} - 2C_{2a} + 12b_{22}C_{2a})}{7 + 3b_2 + 6b_{22} + 6C_{2a} + 6b_2C_{2a} - 4b_{22}C_{2a}}, c_{16} = -\frac{8(-1 + 6b_{22} - C_{2a} + 6b_{22}C_{2a})}{-7 - 3b_2 - 6b_{22} - 6C_{2a} - 6b_2C_{2a} + 4b_{22}C_{2a}}\right\}$$

$$\left\{d_{16} = \frac{8(-C_{2a} + 6b_{22}C_{2a})}{7 + 3b_2 + 6b_{22} + 6C_{2a} + 6b_2C_{2a} - 4b_{22}C_{2a}}, e_{16} = -\frac{2(3 - 18b_{22} - 2C_{2a} + 12b_{22}C_{2a})}{7 + 3b_2 + 6b_{22} + 6C_{2a} + 6b_2C_{2a} - 4b_{22}C_{2a}}\right\}$$

$$\left\{A_1 = -\frac{-2 + 3b_2}{1 + 6b_{22}}, B_1 = -\frac{1 - 3b_2 - 6b_{22}}{1 + 6b_{22}}\right\}$$

$$\left\{A_2 = -\frac{2(-1 + 6b_{22})}{1 + 6b_{22}}, B_2 = -\frac{1 - 18b_{22}}{1 + 6b_{22}}\right\}$$

$$\left\{A_3 = -\frac{-4 + 3b_2 + 12b_{22}}{1 + 6b_{22}}, B_3 = \frac{3(-1 + b_2 + 6b_{22})}{1 + 6b_{22}}\right\};$$

$$\left\{A_4 = -\frac{-7 + 30b_{22}}{1 + 6b_{22}}, B_4 = \frac{6(-1 + 6b_{22})}{1 + 6b_{22}}\right\};$$

$$\left\{A_5 = -\frac{-13 - 3b_2 + 54b_{22}}{1 + 6b_{22}}, B_5 = -\frac{3(4 + b_2 - 20b_{22})}{1 + 6b_{22}}\right\};$$

Dimensional Reduction compatibility fixes the parameters to be:

$$\{b_2 = -18/7, C_{2a} = 16/7, c_{23}a = 16/7\}; b_{22} = (2 + b_2)/8$$

This fixes all the constants. Once again the set of equations is a highly overdetermined set and it is somewhat surprising that a solution exists at all. It would be interesting to find the underlying logic.

Appendix D K-constraints

We derive the K-constraints that occur in the free equations.

For the free part of the equation we do not need the individual $K_{[n]_i;[\bar{m}]_j\mu}$. We can write $\mathcal{L}$ in terms of $Y^{\mu}_{n;\bar{m}}$. Thus the coefficient of $Y^{\mu}_{n;\bar{m}}$ is $\sum_{i,j} K_{[n]_i;[\bar{m}]_j\mu} = \tilde{K}_{n;\bar{m}\mu}$ as defined in (7.1.14).

The general case involves combining the following two terms:

$$\int dz' \int dz'' \dot{G}(z', z'') \frac{\partial}{\partial x_n} \frac{\partial}{\partial Y_m(u)} \delta(u - z') \frac{\partial}{\partial x_{\bar{m}}} \frac{\partial}{\partial Y_{\bar{m}}(u)} \delta(u - z'')\mathcal{L} \ + z' \leftrightarrow z''$$

$$= \int dz' \frac{d}{d\tau} ([\langle Y_n(z')Y_{\bar{m}}(z')\rangle + \langle Y_{\bar{m}}(z')Y_n(z')\rangle](ik_n.ik_{\bar{m}})\mathcal{L} \tag{D.0.1}$$

and

$$\int dz' \dot{G}(z',z'')\frac{\partial^2}{\partial x_n \partial x_{\bar{m}}}\frac{\partial}{\partial Y_{n;\bar{m}}}\delta(u-z')\frac{\partial}{\partial Y}\delta(u-z'')\mathcal{L} \ + z' \leftrightarrow z''$$

$$= \int dz' \frac{d}{d\tau}[\langle Y_{n;\bar{m}}(z')Y(z')\rangle + \langle Y(z')Y_{n;\bar{m}}(z')\rangle](i\tilde{K}_{n;\bar{m}}.ik_0)\mathcal{L} \tag{D.0.2}$$

Now *if*

$$(ik_n.ik_{\bar{m}})\mathcal{L} = (i\tilde{K}_{n;\bar{m}}.ik_0)\mathcal{L} \tag{D.0.3}$$

then we can combine the two terms, (D.0.1) and (D.0.2), and write

$$\int dz' [\frac{d}{d\tau}\frac{\partial^2}{\partial x_n \partial x_{\bar{m}}}\langle Y(z')Y(z')\rangle](-k_n.k_{\bar{m}})\mathcal{L}$$

$$= \int dz' \ \dot{G}(z',z')(-k_n.k_{\bar{m}})\frac{\partial^2}{\partial x_n \partial x_{\bar{m}}}\mathcal{L} \tag{D.0.4}$$

Since the $\tilde{K}_{n;\bar{m}\mu}$ are made of the usual loop variables and no new degrees are involved, the K-constraints (D.0.3) would seem to reduce the number of independent degrees of freedom. However we also have the option of adding *one new* loop variable $k_{n;\bar{m}\mu}$, (with μ chosen to be D, so we can call it $q_{n;\bar{m}}$)to $\tilde{K}_{n;\bar{m}\mu}$ so that the constraint plays the role of determining this variable. $q_{n;\bar{m}}$ should be defined to have the same gauge transformation as $\tilde{K}_{n;\bar{m}\mu}$, viz:

$$q_{n;\bar{m}} \rightarrow q_{n;\bar{m}} + \lambda_p q_{n-p;\bar{m}} + \bar{\lambda}_p q_{n;\bar{m}-p}$$

Then the constraint does not affect the degrees of freedom count.

We have

$$\tilde{K}_{n;\bar{m}\mu} = \bar{q}_n k_{\mu\bar{m}} + \bar{q}_{\bar{m}} k_{n\mu} - \bar{q}_n \bar{q}_{\bar{m}} k_{0\mu}$$

$$\tilde{Q}_{n;\bar{m}} = \frac{q_n}{q_0}q_{\bar{m}} + \frac{q_{\bar{m}}}{q_0}q_n - \frac{q_n}{q_0}\frac{q_{\bar{m}}}{q_0}q_0 + q_{n;\bar{m}} = \frac{q_n q_{\bar{m}}}{q_0} + q_{n;\bar{m}}$$

The constraint (D.0.3) becomes

$$q_{n;\bar{m}} = k_n.k_{\bar{m}} - \bar{q}_n k_{\bar{m}}.k_0 + \bar{q}_{\bar{m}} k_n.k_0 - \bar{q}_n \bar{q}_{\bar{m}} k_0^2$$

thus fixing $q_{n;\bar{m}}$ in terms of the others. This idea was made use of in the discussion on massless spin 2 level.

Appendix E LV to OC map

We give below the relations between the coefficients defining the map from the LV fields to the OC fields that were introduced in Section 8 and reproduced below for convenience.

$$\begin{aligned}
\Phi_{\mu\nu\rho} &= f_1 S_{111\mu\nu\rho} + f_2 S_{3(\mu}\eta_{\nu\rho)} + f_3 S_{A(\mu}\eta_{\nu\rho)} + f_4 p_{(\mu}S_{\nu\rho)} \\
B_{\mu\nu} &= b_1 S_{\mu\nu} + b_2 p_{(\mu}S_{3\nu)} + b_3 p_{(\mu}S_{A\nu)} + b_4 S_3 \eta_{\mu\nu} \\
C_{\mu\nu} &= c_1 A_{\mu\nu} + c_2 p_{[\mu}S_{3\nu]} + c_3 p_{[\mu}S_{A\nu]} \\
A_\mu &= a_1 S_{3\mu} + a_2 S_{A\mu} + a_3 p_\mu S_3
\end{aligned} \tag{E.0.1}$$

They are obtained by requiring that the constraints and gauge transformations map into each other. For the gauge transformation only the tensor and vector parameters were included in the analysis. These are enough to fix D and q_0.

The relation between the gauge parameters in the two formalisms turn out to be:

$$\begin{aligned}
(b_1 + 2b_2)\Lambda_{S\mu} + b_3\Lambda_{A\mu} &= \epsilon_{S\mu} \\
(c_1 + c_3)\Lambda_{A\mu} + 2c_2\Lambda_{S\mu} &= \epsilon_{A\mu} \\
b_1\Lambda_{111\mu\nu} &= \epsilon_{111\mu\nu}
\end{aligned} \tag{E.0.2}$$

We have not worked out the dependence on the scalar parameter in the above. The various equations obtained are:

$$
\begin{aligned}
\frac{-q_0^2 f_1 + f_2(D+2) - q_0^2 f_4)}{12} + \frac{(-b_1 q_0^2 - (b_2+c_2)q_0^2)}{3} + \frac{a_1}{2} &= 0 \\
\frac{f_3(D+2)}{12} + \frac{(-c_1 q_0^2 - (b_3+c_3)q_0^2)}{3} + \frac{a_2}{2} &= 0 \\
\frac{-f_4 q_0}{12} + \frac{(-(b_2-c_2)q_0 + b_4)}{3} + \frac{a_3}{2} &= 0 \\
b_1 q_0^2 - 3a_1 + (b_2 - c_2)q_0^2 &= 0 \\
3a_2 - (b_3 - c_3)q_0^2 + c_1 q_0^2 &= 0 \\
3a_3 + b_4 - (b_2 + c_2)q_0 &= 0 \\
\frac{f_1 q_0^2}{4} - b_1 + \frac{f_4 q_0^2}{4} &= 0 \\
b_3 + \frac{f_3}{4} &= 0 \\
b_4 - \frac{f_2 q_0}{4} &= 0 \\
b_2 + \frac{f_2}{4} - \frac{q_0^2 f_4}{4} &= 0 \\
-3a_3 q_0^2 + 2Db_4 - 3a_1 q_0 - 2b_1 q_0 - 4b_2 q_0 &= 0 \\
3(b_1 + 2b_2) - 2c_2 - 2a_1 &= 0 \\
3b_3 - (c_1 + c_3) - a_2 &= 0 \\
c_1 + c_3 - b_3 - f_3 &= 0 \\
2c_2 + b_1 + 2b_2 - 2f_2 &= 0 \\
f_1 + f_4 - b_1 &= 0
\end{aligned}
\tag{E.0.3}
$$

The solution is

$$
\begin{gathered}
D = 26, q_0 = 2 \\
a_2 = 0, \quad a1 = -((44b_2)/3), \quad b_4 = -2b_2, \quad f_3 = 0, \quad a_3 = (16b_2)/9, \quad f2 = -4b_2, \\
f_1 = -((34b_2)/3), \quad c_1 = -c_3, \quad b_3 = 0, \quad b_1 = -((34b_2)/3), \\
\\
c_2 = (2b_2)/3, \quad f_4 = 0,
\end{gathered}
\tag{E.0.4}
$$

We see that D and q_0 are fixed by this analysis. To fix the remaining coefficients one needs to include the scalar gauge transformation parameters in the analysis.

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